Some more points about lin. sys. of eq^n.

1) We saw in tutorial 2, that if A is a 4×3 matrix, then there exists $b \in \mathbb{R}^4$ s.t. $Ax = b$ is inconsistent.

More generally if A is $m\times n$ with $m > n$, then $\exists\ \vec{b} \in \mathbb{R}^m$ s.t. $A\vec{x} = \vec{b}$ is inconsistent.

Pf: Let $O_1, O_2, \ldots, O_s$, $s \in \mathbb{N}$ be a sequence of row operations that turn A into RREF(A). Then each O_i^{-1} is also a row operation and the sequence $O_s^{-1}, O_{s-1}^{-1}, \ldots, O_1^{-1}$ turns RREF(A) into A

Let $\vec{b}' = \vec{e}_m$ (Recall $\vec{e}_i = (0 \cdots 0\ \underset{\hookrightarrow i\text{-th spot}}{1}\ 0 \cdots 0)^t$) Let $A' = \text{RREF}(A)$.

Note $\text{rank}(A) \leq \min\{m,n\} = n$. So $m > n \Rightarrow$ last Row of A' is zero row.

Therefore $A'\vec{x} = b'$ is inconsistent. Hence $Ax = b$ is inconsistent where b is obtained from b' by performing the sequence of operations $O_s^{-1}, O_{s-1}^{-1}, \ldots, O_1^{-1}$. ∎

2) If A is an $m\times n$ matrix, ~~with~~ the system $Ax = 0$ has a non zero solution $\Longleftrightarrow$ $\text{rank}(A) < n$

In particular if $n > m$ then $\text{rk}(A) \leq m < n$, hence $Ax = 0$ has a non-zero solution.

Pf: If sequence $O_1, O_2, \ldots, O_s$ of row op. turns $A \longrightarrow \text{RREF}(A)$ then the same seq. turns $[A \mid 0]$ to $[\text{RREF}(A) \mid 0]$

If $A' = \text{RREF}(A)$, then we have shown $Ax = 0$ and $A'x = 0$ have same solution set. # d.o.f in solution of $A'x = 0$ is $n - \text{rank}(A')$

(Note $[A' \mid 0]$ is RREF with pivot seq same as that of A')

$\not\exists$ nonzero sol^n iff D.o.F $=0$ ie rank$(A)=n$.

Corollary: If $m \neq n$ and A is $m \times n$ matrix, the linear transf $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is not bijective

Pf: If $m > n$, we saw $\exists\ \vec{b} \in \mathbb{R}^m$ s.t $A\vec{x} = \vec{b}$ is inconsistent. i.e. $\vec{b} \notin \mathrm{Im}(T_A)$. T_A is not onto.

If $n > m$, we saw $A\vec{x} = \vec{0}$ has $\vec{x} = \vec{0}$ as well as $\vec{x} = \vec{x}_* \neq 0$ as solutions. Thus T_A is not 1-1.

Theorem: If T_A is as above, then T_A has inverse iff $\left[\begin{matrix} m = n \ \cancel{\text{and}} \\ = \text{rank}(A) \end{matrix}\right]$

Moreover if inverse exists, it is linear.

Pf: For T_A^{-1} to exist, we need $m = n$, by corollary above.

We know RREF(A) is $\mathbb{1}_n$ (or I_n identity matrix $\left(\begin{smallmatrix}1 & & \\ & \ddots & \\ & & 1\end{smallmatrix}\right)$)

Sol^n of $A\vec{x} = \vec{b}$ are same as those of $I_n \vec{x} = b'$ for some b'

i.e $\vec{x} = b'$ is the only Sol^n. Thus T_A is bijective.

Let $S = T_A^{-1}$ be the inverse function. given $V_1, V_2 \in \mathbb{R}^n$ $\left(\begin{matrix}\text{domain } S \\ = \text{range } T\end{matrix}\right.$

Let $\vec{U_1} = S(V_1)$, $\vec{U_2} = S(V_2)$, i.e $T_A \vec{U_i} = \vec{V_i}$, $i = 1..2$

If $a, b \in \mathbb{R}$ then $T_A(aU_1 + bU_2) = aT_A(U_1) + bT_A(U_2) = aV_1 + bV_2$

Thus $S(aV_1 + bV_2) = aU_1 + bU_2 = aS(V_1) + bS(V_2)$. Thus we have shown S satisfies property $\circledast$ (linearity). Hence it is a lin transf

$\exists$ a matrix B s.t $S(\vec{y}) = B\vec{y}$. Now $T \circ S = id_{\mathbb{R}^n}$, $S \circ T = id_{\mathbb{R}^n}$

implies $AB\vec{x} = \vec{x}$, $BA\vec{x} = \vec{x}$ $\forall\ \vec{x} \in \mathbb{R}^n$

i.e $(AB - I_n)\vec{x} = 0 = (BA - I_n)\vec{x}$ $\forall\ \vec{x} \in \mathbb{R}^n$.

Letting $\vec{x} = \vec{e_j}$, we see jth col of $AB - I_n$, $BA - I_n$ is zero, $j = 1..n$

Thus $AB = BA = I_n$.

The matrix B is called the inverse of matrix A and it is denoted A^{-1}.

Note: The condition $[m=n=\text{Rank}A]$ can also be stated as

$[m=n$ and only solution of $Ax=0$ is $x=0.]$

Let us use this condition to find inverse of $A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ when it exists.

Lemma: Sol^n spc of eq^n $ax+by=0$ where not both a, b are zero. is $\{(x,y)=(-\lambda b, \lambda a) ; \lambda \in \mathbb{R}\}$

Pf: if $(x,y)=(-\lambda b, \lambda a)$ clearly $ax+by = a(-\lambda b)+b(\lambda a)=0$

Sps (x_0, y_0) is a solution. ~~If a=b~~. If $a \neq 0$ we know

$ax_0+by_0=0 \Rightarrow x_0 = \frac{-b}{a} y_0$. Thus $(x_0,y_0)=(-\lambda b, \lambda a)$ for $\lambda = \frac{y_0}{a}$

If $b \neq 0$ $ax_0+by_0=0 \Rightarrow y_0 = -\frac{ax_0}{b}$. Thus $(x_0,y_0)=(-\lambda b, \lambda a)$ for $\lambda = -\frac{x_0}{b}$

$A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$. If $a=b=0$, then $\text{Rank}(A) \leq 1$. So no inverse

So assume one of a, b is nonzero. We need to determine when $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}=\begin{pmatrix} 0 \\ 0\end{pmatrix}$ has only the zero solution $\begin{pmatrix} x \\ y\end{pmatrix}=\begin{pmatrix} 0 \\ 0\end{pmatrix}$.

$ax+by=0 \Rightarrow \begin{pmatrix} x \\ y\end{pmatrix} = \lambda\begin{pmatrix} -b \\ a\end{pmatrix}$. We must find condition s.t.

$(c \ d)\, \lambda \begin{pmatrix} -b \\ a\end{pmatrix}=0$ only if $\lambda=0$. i.e $\lambda(ad-bc)=0$ only if $\lambda=0$

i.e $ad-bc \neq 0$. The inverse matrix $S=[S(e_1) | S(e_2)]$

$S(e_1)=$ Solution $\begin{pmatrix} x \\ y\end{pmatrix}$ of $\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}=\begin{pmatrix} 1 \\ 0\end{pmatrix}$. Set $\begin{pmatrix} x \\ y\end{pmatrix}=+\lambda\begin{pmatrix} -d \\ c\end{pmatrix}$. ~~Need~~

$(a \ b)\, \lambda\begin{pmatrix} -d \\ c\end{pmatrix}=1 \Rightarrow -\lambda(ad-bc)=1 \Rightarrow \lambda = \frac{-1}{ad-bc}$. Thus $S(e_1)=\frac{1}{ad-bc}\begin{pmatrix} d \\ -c\end{pmatrix}$

$S(e_2)=$ by symmetry $(a \ b) \leftrightarrow (c \ d)$ is $\frac{1}{-(ad-bc)}\begin{pmatrix} b \\ -a\end{pmatrix} = \frac{1}{ad-bc}\begin{pmatrix} -b \\ a\end{pmatrix}$ i.e $S=\dfrac{\begin{pmatrix} d & -b \\ -c & a\end{pmatrix}}{ad-bc}$.

Some points about matrix multiplicⁿ

1): If T_A, T_B, T_C are linear transf $\mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^m \xrightarrow{T_B} \mathbb{R}^p \xrightarrow{T_C} \mathbb{R}^q$

then as functions $T_C \circ (T_B \circ T_A) = (T_C \circ T_B) \circ T_A$

Therefore $\vec{x} \mapsto C(BAx)$ equals $\vec{x} \mapsto (CB)Ax$

By uniqueness of matrix of a lin trans

(If $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear and equals $x \mapsto Ax$ as well as $\vec{x} \mapsto B\vec{x}$ then $(A-B)\vec{x} = 0 \ \forall \vec{x}$ take $\vec{x} = e_j$ to conclude j-th col of

$A-B$ is zero $\forall j$ ie $A = B$

Hence $C(BA) = (CB)A$

2) Also as functions: $T_C \circ (T_A + T_B) = T_C \circ T_A + T_C \circ T_B$

Thus $C(A+B) = CA + CB$.

3) $I_m A = A = A I_n$

Pf: Let $id_m : \mathbb{R}^m \to \mathbb{R}^m$ be $\vec{x} \mapsto \vec{x}$.

$id_m \circ T_A = T_A$ and $T_A \circ id_n = T_A$

Thus $I_m A = A$ and $A I_n = A$

$$I_m = \begin{bmatrix} e_1^t \\ e_2^t \\ \vdots \\ e_m^t \end{bmatrix}$$

Q1 What about $R \cdot A$ where R is $m \times m$ matrix

$$\begin{bmatrix} e_1^t \\ \vdots \\ e_{i-1}^t \\ e_j^t \\ e_{i+1}^t \\ \vdots \\ e_{j-1}^t \\ e_i^t \\ e_{j+1}^t \\ \vdots \\ e_m^t \end{bmatrix}$$

A1: RA is A with i-th and j-th Row interchange

Q2: What is RA if $R = \begin{bmatrix} e_1^t \\ \vdots \\ e_{i-1}^t \\ e_i^t + c e_j^t \\ e_{i+1}^t \\ \vdots \\ e_m^t \end{bmatrix}$ $j \neq i$

A2: A with Row $i \mapsto$ Row i + c Row j.

Q3) What is $R \cdot A$ if $R = \begin{bmatrix} e_1^t \\ \vdots \\ e_{i-1}^t \\ c\, e_i^t \\ e_{i+1}^t \\ \vdots \\ e_m^t \end{bmatrix}$, $c \neq 0$

Ans. A with Row $i \mapsto c$ Row i

We call the matrices in Q1, 2, 3 as elementary matrices.

We have just proved: There exists a sequence of Elementary matrices: $E_1, E_2 \dots E_s$ s.t. $E_s E_{s-1} \dots E_1 \cdot A = RREF(A)$

Note: If A is $m \times 1$ matrix i.e. column vector then $A \mapsto RA$ is a lin. transf with matrix R. This lin transf is clearly invertible

Row $i \leftrightarrow$ Row j is its own inverse

Row $i \to$ Row i + c Row j has inverse Row $i \mapsto$ Row i − c Row j.

Row $i \to c$ Row i has inverse Row $i \to$ $1/c$ Row i

Thus E^{-1} is also elem. matrix if E is.

Lemma: If A, B are invertible $m \times m$ matrices then so is AB

moreover $(AB)^{-1} = B^{-1} A^{-1}$

Pf. $\mathbb{R}^m \xrightarrow{T_B} \mathbb{R}^m \xrightarrow{T_A} \mathbb{R}^m$ has inverse $\mathbb{R}^m \xrightarrow{T_A^{-1}} \mathbb{R}^m \xrightarrow{T_B^{-1}} \mathbb{R}^m$

$x \longmapsto AB\,x$ $\qquad$ $x \longmapsto B^{-1} A^{-1} x$.

Thm: An $n \times n$ matrix A is invertible

$\iff$ A is a product of $n \times n$ elem. matrices.

Proof: $\Leftarrow$ If $E = E_s E_{s-1} \cdots E_1$, then by previous lemma $E^{-1} = E_1^{-1} E_2^{-1} \cdots E_s^{-1}$ exists

$\Rightarrow$ Since A is $n \times n$ of rank n, $RREF(F)(A) = \mathbb{I}_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

We have: $E_s E_{s-1} \cdots E_1 A = I_n$

$\Rightarrow$ Multiplying both sides by $(E_s E_{s-1} \cdots E_1)^{-1}$, we get

$A = E_1^{-1} E_2^{-1} \cdots E_s^{-1}$ a product of elementary matrices ∎

(Note: $A^{-1} = E_s E_{s-1} \cdots E_1$!)

How to determine A^{-1} when A is invertible.

Consider matrix $[A \mid I_n]$ of size $n \times 2n$

If $E_s E_{s-1} \cdots E_1 A = \mathbb{I}_n$ (by Row operations) then

$E_s E_{s-1} \cdots E_1 [A \mid I_n] = [I_n \mid E_s E_{s-1} \cdots E_1]$

$= [I_n \mid A^{-1}]$

Procedure: Bring $[A \mid I_n]$ to RREF to get $[\mathbb{I}_n \mid A^{-1}]$.
(assuming A^{-1} exists)