MTH-201, Linear Algebra   Lec 1,   Aug 3 -2015

Linear Algebra: Study of "Vector spaces" and "linear transformations" between vector spaces.

It is a subject of great importance in Science and Technology.
For example:

1) Google's PageRank algorithm can be completely understood using this course.

2) L.A is at the foundation of the framework of Quantum Mechanics.
 - Observable quantities are represented as "Hermitian" linear transformations from a vector space V (Hilbert spc) to V.
 - measurement of this quantity yields an eigenvalue of this lin. transf and "state of the system" collapses to ~~the~~ an eigenvector corresponding to this eigenvalue.

Def^n: An $m \times n$ matrix A is an array with m rows and n columns. The entry in the i-th row and j-th column is denoted as A_{ij}.

Ex: $A = \begin{bmatrix} 9 & 8 & 1 & 8 \\ 7 & 3 & 8 & 4 \\ 4 & 5 & 7 & 7 \end{bmatrix}$ is a 3×4 matrix

$A_{13} = 1$ and $A_{22} = 3$ etc.

[PS: entries are in a field say $\mathbb{R}$ (real numbers) or $\mathbb{C}$ (complex numbers)]

We can add two matrices of the same ~~3~~ size $m \times n$:
Just define $(A+B)_{ij} = A_{ij} + B_{ij}$.

We can scalar multiply a matrix A by a scalar c (from our field)
$(cA)_{ij} = c\, A_{ij}$

Def^n If $m=1$, a $m \times n$ matrix is a <u>row vector</u>
If $n=1$ a $m \times n$ matrix is a <u>column vector</u>.

Def^n: <u>Transpose</u>: If A is a $m \times n$ matrix then A^t is $n \times m$ matrix
$(A^t)_{ij} = A_{ji}$.

Transpose turns a row vector of length n into a column vector of length n and vice-versa.

Def^n given ~~a~~ $1 \times m$ matrix A and $m \times 1$ matrix B
Define AB to be the scalar $a_1 b_1 + \cdots + a_m b_m$ where

$$A = [a_1, a_2, \ldots, a_m], \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

In other words AB is the "dot product" of the vectors A^t and B.

Def^n: given $m \times n$ matrix A and $n \times p$ matrix B

Define AB to be an $m \times p$ matrix whose (i,j)-th entry is the ~~A~~ dot product of the i-th row of A (thought of as a column vector) and the j-th column of B

$$A = \underset{2\times 4}{\begin{bmatrix} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 1 \end{bmatrix}}, \quad B = \underset{4\times 3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix}}$$

then AB is a 2×3 matrix

$$AB = \begin{bmatrix} 4 & 8 & 0 \\ 12 & 18 & 1 \end{bmatrix}$$

<u>Formula</u>: $$(AB)_{ij} = \sum_{l=1}^{n} A_{il} B_{lj}.$$

Linear equations and Gauss-Jordan (Gaussian) elimination.

given m eq^n (linear) in n-unknowns: we associate a $m\times(n+1)$ matrix as follows.

$$\begin{aligned} 2x_1 + 4x_2 - 2x_3 + 2x_4 + 4x_5 &= 2 \\ x_1 + 2x_2 - 1x_3 + 2x_4 + 0x_5 &= 4 \\ 3x_1 + 6x_2 - 2x_3 + 1\cdot x_4 + 9x_5 &= 1 \\ 5x_1 + 10x_2 - 4x_3 + 5x_4 + 9x_5 &= 9 \end{aligned} \longleftrightarrow \left[\begin{array}{ccccc|c} 2 & 4 & -2 & 2 & 4 & 2 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array}\right]$$

$$A\begin{pmatrix} x_1 \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_m \end{pmatrix} \longleftrightarrow [A \mid b]$$

3 operations (which don't change solution set)

(1): multiply an eq^n by non-zero scalar
(2) Add a multiple of an eq^n to a different eq^n:
(3) interchange order of two equations

$\rightarrow$ multiply a row by non-zero scalar
Row $i \mapsto$ Row i + $c\cdot$ Row j.
interchange two rows of the matrix

(1) and (3) are obvious. How about (2)? ~~→~~
If $(x_1 \cdots x_n)$ is a Solution of original System then it is also a solution of the modified system.
Note that original system is obtained from modified system by doing Row $i \mapsto$ Row i $-$ c Row j !
Thus solution of modified system are also Solution of original System

Q: Using these 3 elementary row operations how much can we simplify the matrix / system of eq^n

A: RREF matrix: Reduced Row Echelon form.
We need to consider matrices in RREF only.

Def^n: An $m \times n$ matrix A is said to be RREF if

There is a sequence $1 \le p_1 < p_2 < \cdots < p_r \le n$ with $r \le m$

such that

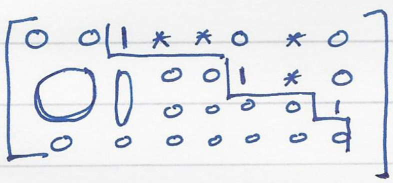
For each i:
$$A_{ij} = \begin{cases} 0 & \text{if } j < p_i \\ 0 & \text{if } j = p_k \text{ for } k > i \\ 1 & \text{if } j = p_i \\ 0 & \text{if } i > r \end{cases}$$

In other words the i-th row has zeros for the first $p_i - 1$ entries

The p_i-th entry is 1.

The p_k-th entry is 0 if $k > i$

The rows from $r+1$ to m are zero rows.



$(p_1, p_2, \ldots, p_r)$ is called PIVOT SEQUENCE.

r is called rank of matrix A

Note: $r \le m$, $r \le n$ implies $r \le \min\{m, n\}$.

Ex:
$$\left[\begin{array}{ccccc|c} 2 & 4 & -2 & 2 & 4 & 2 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array}\right] \xrightarrow{R_1 \to R_1/2} \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 2 & 1 \\ 1 & 2 & -1 & 2 & 0 & 4 \\ 3 & 6 & -2 & 1 & 9 & 1 \\ 5 & 10 & -4 & 5 & 9 & 9 \end{array}\right] \xrightarrow[R_4 \mapsto R_4 - 5R_1]{R_2 \mapsto R_2 - R_1,\ R_3 \mapsto R_3 - 3R_1} \left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array}\right]$$

$\downarrow$ ~~$R_1 \mapsto R_1 + R_3$~~, ~~$R_4 \mapsto R$~~, $R_2 \leftrightarrow R_3$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 0 & -1 & 4 \end{array}\right] \xrightarrow[R_4 \mapsto R_4 - R_2]{R_1 \mapsto R_1 + R_2} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -1 & 5 & \cancel{3}\,-1 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 2 & -4 & 6 \end{array}\right] \xrightarrow{R_4 \mapsto R_4 - 2R_3} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -1 & 5 & \cancel{3}\,-1 \\ 0 & 0 & 1 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\downarrow$

$R_1 \mapsto R_1 + R_3$

$R_2 \mapsto R_2 + 2R_3$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Pivot seq is $1, 3, 4$ and we have

$$\begin{aligned} x_1 &= 2 - 2x_2 - 3x_5 \\ x_3 &= 4 + x_5 \\ x_4 &= 3 + 2x_5 \end{aligned}$$

Solution set $= \left\{ \begin{pmatrix} 2 \\ 0 \\ 4 \\ 3 \\ 0 \end{pmatrix} + a \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} -3 \\ 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} : a, b \in \mathbb{F} \right\}$

The modified system is $A'x = b'$, with $[A' \mid b']$ in RREF

Note A' is also in RREF.

(*) If $(p_1, p_2, \ldots, p_r)$ is the pivot sequence of $[A' \mid b']$ and $\boxed{p_r = n+1}$

then the r-th eq^n is of the form $0 = 1$ → No Solution.

(*) $\boxed{\text{If } p_r \leq n}$, then the pivot sequence of A' is $(p_1, p_2, \ldots, p_r)$ again

The number of non pivot variables (non pivot columns of A') is $n-r$

and the Solution space has $n-r$ "degrees of freedom"
(if $n=r$, there is a unique solution)

The non-pivot variables are free to be what they want.
The pivot variables are completely pinned down.