MTH201: EXTRA PROBLEMS FOR WEEK 1

(1) What is the maximum number of non-pivot and non-zero entries possible for a RREF matrix of size $m \times n$ and rank r with pivot sequence $(p_1, p_2, \ldots, p_r)$.

For which pivot sequence (with r fixed), is the above number maximum/minimum ?

(2) Given constants a and b, for which scalars c is the vector $(1, c, c^2)^t$ a linear combination of the vectors $(1, a, a^2)^t$ and $(1, b, b^2)^t$.
(Recall a linear combinations of vectors $\vec{u}$ and $\vec{v}$ is a vector of the form $a\vec{u} + b\vec{v}$ where a and b are scalars.)

(3) Find the polynomial $p(t) = a + bt + ct^2 + dt^3$ whose graph goes through the points $(0, 1), (1, 0), (-1, 0)$ and $(2, -15)$. Your solution must involve bringing a 4×5 augmented matrix into RREF.

1. Answers

(1)

$$(n - j_1 + 1 - \ell) + (n - j_2 + 2 - \ell) + \cdots + (n - j_\ell + \ell - \ell) = \ell(n - \ell) - \sum_{i=1}^{\ell} (j_i - i).$$

This attains the maximum value $\ell(n - \ell)$ when the pivot vector is $\vec{j} = (1, 2, \ldots, \ell)$, and minimum value 0, when $\vec{j} = (n - \ell + 1, \ldots, n)$.

(2) When $c = a$ or $c = b$.

(3) Consider the linear system $Ax = B$ where $x = (a, b, c, d)^t$ and $B = (1, 0, 0, -15)^t$ and $A_{ij} = t_i^{j-1}$ where $(t_1, t_2, t_3, t_4) = (0, 1, -1, 2)$. Solve it by Gaussian elimination. $p(t) = (1 - t^2)(2t + 1)$