Mid-Term Review

Week 1 & 2: RREF, Gauss elimination, matrix mult.

$A_{m\times n} \longrightarrow RREF(A) = A'$ (it is unique).

$A' = PA$ where $P_{m\times m}$ is an invertible matrix. It is a product of Elementary $m\times m$ matrices.

(Conversely: Every $m\times m$ invertible matrix is product of elem matrices)

Pivot sequence $1 \le p_1 < p_2 \quad < p_r \le n$

where $r \le \min\{m, n\}$ is defined to be rank of A.

1) Rows $r+1$ to m (if any) are zero
2) all entries in i-th row before p_i-th col are zero.

   p_i-th entry of i-th row is 1
3) p_j-th Col is e_j.

$A\vec{x} = \vec{b}$ Same Sol^n as $A'x = b'$ where $[A'|b']$ obtained by row operations on $[A|b]$ for example:

$[A'|b'] = RREF[A|b]$. If $[A'|b'] = RREF[A|b]$ then

$A' = RREF(A)$

Rank [A | b] = Rank [A]

$\iff$ no pivot of [A' | b'] in last column

$\iff$ $Ax = b$ is consistent.

In this case Solutions exist.

If x_* is any Solution then the Solution set is

$$x_* + \mathrm{Ker}(A) = x_* + \mathrm{Ker}(A')$$

d.o.f of Sol^n of $Ax=b$ (if consistent)

is nullity (A).

We showed $\mathrm{Ker}(A')$ has same dimension as $\mathrm{Ker}(A'P)$

where $P_{n\times n}$ is invertible. We can ~~show~~ P s.t $A'P = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$

So $\mathrm{Ker}(A'P) = \mathrm{Span}\{e_{r+1}, \dots e_n\}$ which is $n-r$ dim'l.

(Lemma: If $\mathbb{R}^m \xrightarrow{T} \mathbb{R}^n$ or $V \xrightarrow{T} V$ is isom
then $\dim(T(W)) = \dim(W)$.)

~~There is a basis of $\mathrm{Ker}(A)$ of the form $\{V_1 \dots V_{n-r}\}$~~

~~where Let $1 \le q_1 < q_2 \dots < q_{n-r} \le n$ be non-pivot Sequence.~~

~~$V_j, q_i = \delta_{ij}$~~

Pivot Variables are completely determined in term of non pivot Variables.

$Ax=0$ has unique Sol^n (namely $\vec{x}=\vec{0}$)

$\iff n-r=0$. rank is n

$\iff$ Cols of A are lin indep in $\mathbb{R}^m$

Claim 1) $M>m$ vectors in $\mathbb{R}^m$ are always dependent.

Pf: otherwise $\underset{\cancel{M\times m}\ m\times M}{\begin{bmatrix} V_1 \cdots V_M \end{bmatrix}}$ matrix has rank $M>m$ $\to$ $\Rightarrow\!\Leftarrow$

Claim 2) Any set of $M<m$ vectors in $\mathbb{R}^m$ $\{v_1 \cdots v_M\}$ can be enlarged to form a basis $\{v_1 \cdots v_M\, v_{M+1} \cdots v_m\}$

Let $A=[V_1 \cdots V_M]_{m\times M}$. Pick $Q_{m\times m}$ s.t

$$QA=\begin{bmatrix} I_M \\ 0 \end{bmatrix}=RREF(A)$$

$$\text{Let } B=Q^{-1}\left[\begin{array}{c|c} I_M & 0 \\ 0 & I_{m-M} \end{array}\right]=\left[V_1 \cdots V_M \,|\, V_{M+1} \cdots V_m\right].$$

Claim 3) If $\{v_1 \cdots v_M\}\subset\mathbb{R}^m$ span $\mathbb{R}^m$, then a subset of $\{v_1 \cdots v_M\}$ forms basis of $\mathbb{R}^m$.

Ans. Let $A=[V_1 \cdots V_M]_{m\times M}$ Let $Q_{m\times m}\, A=A'=RREF(A)$

~~$Im(QA)=Q\cdot Im A=Q\cdot\mathbb{R}^m=\mathbb{R}^m$~~ ~~Basis for $Im A=\mathbb{R}^m$ is PIVOT Cols of A (not A')~~ Q.E.D. ∎

Q: What is $\dim$ of RREF matrix? $A_{m\times n}$

A: $\mathbb{R}^r \subset \mathbb{R}^m$ sitting as $\{(x_1 \dots x_r, \underbrace{0 \dots 0}_{m-r}) : x_i \in \mathbb{R}\}$

Thus returning to our problem $\overset{(\text{Claim 3})}{\wedge}$ $r = m$.

Q: What is a basis of $A_{m\times n}$? in rel^n to RREF(A).

A: Columns $p_1 \dots p_r$ of $\underline{\underline{A}}$ where $p_1 - p_r$ is pivot seq of A'

So Cols $p_1 \dots p_m$ of A i.e.

$\{V_{p_1} \dots V_{p_m}\} \subset \{V_1 \dots V_M\}$ form basis of $\mathbb{R}^m$.

Change of Basis.

$$\mathbb{R}^n \xrightarrow{T_A} \mathbb{R}^m$$
$$x \longmapsto Ax = y.$$

New Coords on $\mathbb{R}^n$	New coor on $\mathbb{R}^m$
$x' = Px$, $P_{n\times n}$ invertible	$y' = Qy$, $Q_{m\times m}$ invertible
New basis of $\mathbb{R}^n$ $\{f_1 \dots f_n\} = $ Cols of P^{-1}	New basis of $\mathbb{R}^m$ $\{g_1 \dots g_m\} = $ Cols of Q^{-1}

$$\left[\, Q^{-1}y' = y = Ax = AP^{-1}x' \,\right] \implies y' = QAP^{-1}x$$

j-th col of $QAP^{-1} = \begin{pmatrix} c_1 \\ c_m \end{pmatrix} \iff Af_j = c_1 g_1 + \dots c_m g_m$

If $A: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and same basis on source and target. Then. If new basis of $\mathbb{R}^n$ = cols of $P^{-1}_{n\times n}$ and new co-ords $x' = Px$, the new matrix is PAP^{-1}.

Def If A, B are $n\times n$ matrices, ~~and $P_{n\times n}$ is invertible~~ we say B is <u>similar</u> to A if $\exists$ invertible $n\times n$ matrix P s.t $B = PAP^{-1}$.

<u>Question</u>: Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ ~~$\vec{x} \mapsto \vec{u}\times\vec{x}$~~ $\vec{x} \mapsto uu^t\vec{x}$. where $\|u\| = 1$.

Sps $\|v\| = 1$, $u \perp v$. Let $w = \vec{u}\times\vec{v}$. Let $P^{-1} = [u|v|w]$.

Find PAP^{-1} where $A = uu^t$ by

1) Calculating PAP^{-1} (note $P^{-1} = P^t$ here)
2) interpret^n of j-th col of P^{-1} = T(j-th new basic vectors) ~~exp~~ coords wrt new basis.