Q1:

Augmented matrix
$$\left[\begin{array}{ccc|c} 0 & 1 & 2k & 0 \\ 1 & 2 & 6 & 2 \\ k & 0 & 2 & 1 \end{array}\right]$$

$\downarrow R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 6 & 2 \\ 0 & 1 & 2k & 0 \\ k & 0 & 2 & 1 \end{array}\right] \xrightarrow{R_3 \mapsto R_3 - kR_1} \left[\begin{array}{ccc|c} 1 & 2 & 6 & 2 \\ 0 & 1 & 2k & 0 \\ 0 & -2k & 2-6k & 1-2k \end{array}\right]$$

$\downarrow$ $R_1 \mapsto R_1 - 2R_2$, $R_3 \mapsto R_3 + 2kR_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 6-4k & 2 \\ 0 & 1 & 2k & 0 \\ 0 & 0 & 4k^2-6k+2 & 1-2k \end{array}\right] \longrightarrow$$

If $k = 1/2$: $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

If $k = 1$: $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]$

$k \notin \{1, 1/2\}$ $\xrightarrow[R_3/4k^2-6k+2]{R_3 \mapsto}$ $\left[\begin{array}{ccc|c} 1 & 0 & 6-4k & 2 \\ 0 & 1 & 2k & 0 \\ 0 & 0 & 1 & \frac{1}{2-2k} \end{array}\right]$

$\underline{\underline{2}}$

$4k^2 - 2k - 4k + 2$

$4k(k-1) - 2(k+1)$

$2k(k-2)$

$k = \frac{6 \pm \sqrt{36-32}}{2} = \frac{6 \pm 2}{2} = 4, 2$

Pivot seq of Aug. matrix
$$\begin{cases} (1,2),\ r=2,\ p_r \neq 4 & \text{if } k = 1/2 \\ (1,2,4)\ r=3\ p_r = 4 & \text{if } k=1 \\ (1,2,3)\ r=3,\ p_r \neq 4 & \text{if } k \notin \{1, 1/2\} \end{cases}$$

b) no Solutn in Case $k=1$ as $p_r = n+1$.

a) # d.o.f $= n - r = 0$ in Case $k \notin \{1, 1/2\}$ $\longrightarrow$ unique Sol^n

c): # d.o.f $= n - r = 1$ in Case $k = 1/2$ $\longrightarrow$ Infinitely many Sol^n.

Q2

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \mapsto R_3 - R_1]{R_2 \mapsto R_2 - R_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\downarrow \begin{array}{l} R_1 \mapsto R_1 - 2R_2 \\ R_3 \mapsto R_3 + R_2 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 11 & -6 & -4 \\ 0 & 1 & 0 & -3 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right] \xleftarrow[R_2 \to R_2 + R_3]{R_1 \mapsto R_1 - 4R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 3 & -2 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 \end{array}\right]$$

Thus inverse is $\begin{bmatrix} 11 & -6 & -4 \\ -3 & 2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

Q3

Solution of this system are $\left\{ \begin{pmatrix} a \\ b \\ -2a-3b \end{pmatrix} : a, b \in \mathbb{R} \right\}$

$= \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} \right\}$

Note $a\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ -2a-3b \end{pmatrix} = \vec{0} \iff a = b = 0$

So the vectors are lin. indep.

Thus Basis for Solution spc of $2x_1 + 3x_2 + x_3 = 0$

is $\left\{ \begin{matrix} 1 \\ 0 \\ -2 \end{matrix}, \begin{matrix} 0 \\ 1 \\ -3 \end{matrix} \right\}$.

Q4

a) $T(af+bg) = \int_0^1 af(x)+bg(x)\,dx = a\int_0^1 f + b\int_0^1 g$ (proved in MTH-101)

$= aT(f) + bT(g)$. Thus T is linear.

b) Take $f(x)=1$ (const polynomial)

We see $Tf = \int_0^1 dx = 1$. Since $\{1\}$ is a basis of $\mathbb{R}$

we see $Im(T) = \mathbb{R}$ hence $rk(T)=1$

Basis for $Im(T)$ is $\{1\}$.

~~By~~

c) By Rank-Nullity theorem $3 = rk(T) + nullity(T)$

$\Rightarrow nullity(T) = 2$.

d) Sps $p = a_0 + a_1x + a_2x^2 \in \ker T$, ie

$a_0 + \frac{a_1}{2} + \frac{a_2}{3} = 0$ ie $p(x) = a_1(x - 1/2) + a_2(x^2 - 1/3)$

ie $\{x - 1/2,\ x^2 - 1/3\}$ spans $\ker(T)$.

If $q(x) = a(x - 1/2) + b(x^2 - 1/3) = 0$ with not a, b both zero

then $\deg(q) \le 2$, $q \ne 0$ has more than 2 roots $\rightarrow$ ↯

Thus: $x - 1/2$, $x^2 - 1/3$ are lin indep.

Thus $\{x - 1/2,\ x^2 - 1/3\}$ is a basis for $\ker(T)$.

Q5

Pf1:
Let $A = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix}_{n \times m}$. We are given.

$$(A^t A)_{ij} = v_i^T v_j = \delta_{ij} \quad \text{ie} \quad A^t A = I_m$$

If $\sum_{i=1}^{m} c_i v_i = 0$ then $A\begin{pmatrix} c_1 \\ c_m \end{pmatrix} = 0 \Rightarrow A^t A \begin{pmatrix} c_1 \\ c_m \end{pmatrix} = 0$

$$\Downarrow$$

$$\begin{pmatrix} c_1 \\ c_m \end{pmatrix} = 0$$

Thus $\{v_1 - v_m\}$ are lin. indep.

Pf2: Sps $\sum_{i=1}^{m} c_i v_i = \vec{0}$. Taking dot product with v_j we get $c_j (v_j \cdot v_j) + \sum_{i \neq j} c_i (v_i \cdot v_j) = c_j = 0$

Since j was arbitrary. We see all c_i are zero.
Thus $\{v_1 \cdots v_m\}$ are linearly independent.

Q6

a) False $(\operatorname{Ker} A' = \operatorname{Ker} A$ always$)$

b) True Sps $Av \in \operatorname{Ker} A$ then $A^2 v = 0$ but $A = A^2 \Rightarrow Av = 0$.

c) True $A^2 = 0 \Rightarrow A(\operatorname{Im} A) = 0 \Rightarrow \operatorname{Im} A \subset \operatorname{Ker} A \Rightarrow rk(A) \leq \text{nulldim}(A)$

Thus $n = rk(A) + \text{nulldim}(A) \geq 2rk(A)$

$\Rightarrow rk(A) \leq n/2 = 5$

d) True If $A^2 v = b$ then $Aw = b$ for $w = Av$.

Q8)

a) True. If A, I are dependent then $A = \lambda I$ and $\operatorname{Ker} S_A = M = 4$ dim'l

If A, I are indep then $\operatorname{Ker} S_A$ contains $\operatorname{Span}\{A, I\}$ so $\operatorname{Ker} S_A \geq 2$ dim'l

b) True $\exists\ Q_{m\times m}\ P_{n\times n}$ invertible s.t. $QAP = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

$r = rk(A)$

Thus $P^t A^t Q^t = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ Since left (right) mult by invertible matrices does not change rank we see

$rk(A^t) = rk\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = r.$

Q 7

Proof:

Let $\vec{w}$ be a vector (non zero) perpendicular to given Plane. P. ~~Pick linearly independent~~ We may assume $\vec{w}$ is a unit vector. Pick a unit vector $\vec{u}$ in the plane, and set $\vec{v} = \vec{u} \times \vec{w}$. Certainly $\vec{v} \in P$.

By problem 5, $\vec{u}, \vec{v}, \vec{w}$ are 3 lin. indep vectors in $\mathbb{R}^3$ and hence form a basis of $\mathbb{R}^3$

(Note $\vec{v}, \vec{u} \in P \Rightarrow \vec{v}, \vec{u}$ are $\perp$ to $\vec{w}$ and $\vec{v} \cdot \vec{u} = \cancel{\vec{v} \cdot \vec{u}}$
$= (\vec{u} \times \vec{w}) \cdot \vec{u} = 0$).

The matrix of $\vec{x} \mapsto A\vec{x}$ wrt basis $\{\vec{u}, \vec{v}, \vec{w}\}$ is QAQ^{-1} where $Q^{-1} = [\vec{u} | \vec{v} | \vec{w}]$ (proved in class)

j^{th} col of QAQ^{-1} is coordinates wrt $\{\vec{u}, \vec{v}, \vec{w}\}$ of

$$\begin{Bmatrix} A\vec{u} & \text{if } j=1 \\ A\vec{v} & \text{if } j=2 \\ A\vec{w} & \text{if } j=3 \end{Bmatrix} = \begin{Bmatrix} \vec{u} & \text{if } j=1 \\ \vec{v} & \text{if } j=2 \\ -\vec{w} & \text{if } j=3 \end{Bmatrix}$$

i.e $QAQ^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

MTH201 Mid Term Examination 22/09/2015, 2 hours, Max. Marks: 32

Instructions: Double check your work for calculation errors. Each question is worth 4 marks. Question 6 has negative marking. *No Calculators allowed.*

Q 1) Consider the equations $\left|\begin{array}{llll} & y & +2kz & =0 \\ x & +2y & +6z & =2 \\ kx & & +2z & =1 \end{array}\right|$, where k is an arbitrary constant.

a) For which values of the constant k does this system have a unique solution?
b) When is there no solution?
c) When are there infinitely many solutions?

Q 2) Use row reduction to find inverse of the matrix $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$.

Q 3) Find a basis for the subspace of $\mathbb{R}^3$ defined by $2x_1 + 3x_2 + x_3 = 0$.

Q 4) Let $\mathcal{P}_2$ denote the 3-dimensional vector space of polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ of degree atmost 2. Let $T : \mathcal{P}_2 \to \mathbb{R}$ be the function given by $T(f) = \int_0^1 f(x)dx$.

a) Show that T is a linear transformation
b) Find a basis (with reasoning) for image of T, and determine rank(T).
c) Determine the nullity of T.
d) Find a basis (with reasoning) for kernel of T.

Q 5) Consider some mutually perpendicular unit vectors $v_1, v_2, \ldots, v_m$ in $\mathbb{R}^n$. Show that these vectors are necessarily linearly independent.

Q 6) Answer as T/F (without reasoning). $+1$ *if correct,* -1 *if incorrect,* 0 *if unattempted*

a) If $A' = \text{RREF}(A)$, then $\dim(\ker(A')) = \dim(\ker(A))$, but $\ker(A')$ need not equal $\ker(A)$.
b) If A is any $n \times n$ matrix such that $A^2 = A$, then $\text{im}(A)$ and $\ker(A)$ have only the zero vector in common.
c) If $A^2 = 0$ for a 10×10 matrix A, then $\text{rank}(A) \leq 5$.
d) If the linear system $A^2x = b$ is consistent, then the system $Ax = b$ must be consistent as well.

Q 7) Recall that square matrices A and B are called similar if there exists an invertible matrix P such that $B = PAP^{-1}$.
Prove or disprove: If A is a 3×3 real matrix such that $x \mapsto Ax$ represents reflection in a plane, then A is similar to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Q 8) a) Let M be the 4-dimensional vector space of all 2×2 real matrices. For $A \in M$, let $S_A : M \to M$ be the linear transformation given by $S_A(X) = AX - XA$.
Prove or disprove: There exists $A \in M$ such that $\ker(S_A)$ is one dimensional.

b) *Prove or disprove*: For any $m \times n$ real matrix A, $\text{rank}(A) = \text{rank}(A^t)$ (where A^t is the transpose of A).