MTH 201 – ASSIGNMENT 6

In this assignment $\mathcal{P}_m = \{\sum_{i=0}^m a_i x^i : a_i \in \mathbb{R}\}$.

(1) Find the matrix A' of the linear transformation $T(x) = Ax$ with respect to the basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$.

a) $A = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 6 \end{smallmatrix}\right]$; $\vec{v}_1 = \left[\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right]$, $\vec{v}_2 = \left[\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right]$, b) $A = \left[\begin{smallmatrix} 13 & -20 \\ 6 & -9 \end{smallmatrix}\right]$; $\vec{v}_1 = \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$, $\vec{v}_2 = \left[\begin{smallmatrix} 5 \\ 3 \end{smallmatrix}\right]$

(2) Are the following subsets of $\mathcal{P}_2$ subspaces of $\mathcal{P}_2$. If so find the dimension by finding a basis.

$$a)\ \{p(x) : p'(1) = p(2)\}, \qquad b)\ \{p(x) : \int_0^1 p(x)dx = 0\}.$$

(3) Let V be the vector space of all 2×2 real matrices. Find the dimension of the following subspaces by finding a basis:

$$a)\ \{A : A\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] = 0_{2\times 2}\}, \quad b)\ \{A : \left[\begin{smallmatrix} 1 & 2 \\ 3 & 6 \end{smallmatrix}\right] A = 0_{2\times 2}\}.$$

(4) Find the kernel, nullity, rank and image (in this order) of the linear transformation from $\mathcal{P}_2 \to \mathbb{R}^2$ given by $p(x) \mapsto (p(7), p(11))$.

(5) Let V denote the vector space of infinite sequences of real numbers. We may also think of V as the vector space of functions from $\mathbb{N}$ to $\mathbb{R}$. Let $\mathcal{P}$ denote the vector space of all polynomials with real coefficients. Find the image and kernel of the linear transformation

$$T : \mathcal{P} \to V \text{ given by } p(x) \mapsto (p(0), p'(0), p''(0), \dots)$$

(6) (*) Consider the basis $\{1, x, x^2/2, x^3/6, \dots, x^m/m!\}$ for $\mathcal{P}_m$. Consider the following functions from $\mathcal{P}_m$ to itself, where a is a fixed real number:

$$T(p(x)) = p'(x), \qquad S(p(x)) = p(x+a).$$

You may assume $m = 2$ below except in the parts f) and g).

(a) Prove that T and S_a are linear transfromations.
(b) Find the matrix of T wrt the chosen basis.
(c) Let $m = 2$. Find the matrix of S_a wrt the chosen basis. Guess/find the matrix of S_a for general m.
(d) Let $m = 2$. Show that $T^i = 0$ if $i > 2$.
(e) Using the previous part, it makes sense to talk about

$$\exp(aT) = I + aT + a^2T^2/2 + \cdots + a^iT^i/i! + \dots$$

What is the matrix of $\exp(aT)$ wrt the chosen basis ?
(f) Guess/find the answers to the previous parts for general m.
(g) Relate the transformations $\exp(aT)$ and S_a. What's the interpretation ? ⌣̈