MTH 201 – ASSIGNMENT 5

(1) Find a basis for image and kernel of T_A for:

$$\text{a) } A = \begin{bmatrix} 1 & 3 & 9 \\ 4 & 5 & 8 \\ 7 & 6 & 3 \end{bmatrix}, \qquad \text{b) } A = \begin{bmatrix} 2 & 4 & 8 \\ 4 & 5 & 1 \\ 7 & 9 & 3 \end{bmatrix}$$

(2) For which values of the constant k do the vectors $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 2 & 3 & 4 & k \end{bmatrix}$ form a basis of $\mathbb{R}^4$?

(3) Consider some mutually perpendicular unit vectors $v_1, v_2, \ldots, v_m$ in $\mathbb{R}^n$. Show that these vectors are necessarily linearly independent.

(4) a) A subspace W of $\mathbb{R}^n$ is called a hyperplane if W is defined by a homogeneous linear equation $c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0$, where at least one of the coefficients c_i is nonzero. What is the dimension of a hyperplane in $\mathbb{R}^n$? Justify your answer carefully. What is a hyperplane in $\mathbb{R}^3$? What is it in $\mathbb{R}^2$?

b) Consider a nonzero vector $v \in \mathbb{R}^n$. What is the dimension of the space (denoted $v^{\perp}$) of all vectors in $\mathbb{R}^n$ that are perpendicular to v?

(5) Determine whether the vector $\vec{x}$ is in the span W of the vectors $v_1, \ldots, v_m$. (Proceed "by inspection" if possible, and use RREF form if necessary). If $\vec{x}$ is in W, find the coordinates $[\vec{x}]_{\mathcal{B}}$ of $\vec{x}$ with respect to the basis $\mathcal{B} = (v_1, \ldots, v_m)$.

a) $\vec{x} = \begin{bmatrix} 7 \\ 16 \end{bmatrix}$; $\vec{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$, b) $\vec{x} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$; $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$

(6) find the matrix B of the linear transformation $T(x) = Ax$ with respect to the basis $\mathcal{B} = (\vec{v}_1, \vec{v}_2)$.

a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$; $\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, b) $A = \begin{bmatrix} 13 & -20 \\ 6 & -9 \end{bmatrix}$; $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

(7) Do the even numbered problems from 1-38 in the following list of T/F Questions. Supply reasons, not just T/F.

TRUE OR FALSE?

1. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_m$ are any two bases of a subspace V of $\mathbb{R}^{10}$, then n must equal m.
2. If A is a 5×6 matrix of rank 4, then the nullity of A is 1.
3. The image of a 3×4 matrix is a subspace of $\mathbb{R}^4$.
4. The span of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ consists of all linear combinations of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.
5. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent vectors in $\mathbb{R}^n$, then they must form a basis of $\mathbb{R}^n$.
6. There exists a 5×4 matrix whose image consists of all of $\mathbb{R}^5$.
7. The kernel of any invertible matrix consists of the zero vector only.
8. The identity matrix I_n is similar to all invertible $n \times n$ matrices.

9. If $2\vec{u}+3\vec{v}+4\vec{w}=5\vec{u}+6\vec{v}+7\vec{w}$, then vectors $\vec{u}, \vec{v}, \vec{w}$ must be linearly dependent.

10. The column vectors of a 5×4 matrix must be linearly dependent.

11. If matrix A is similar to matrix B, and B is similar to C, then C must be similar to A.

12. If a subspace V of $\mathbb{R}^n$ contains none of the standard vectors $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$, then V consists of the zero vector only.

13. If vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ are linearly independent, then vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ must be linearly independent as well.

14. The vectors of the form $\begin{bmatrix} a \\ b \\ 0 \\ a \end{bmatrix}$ (where a and b are arbitrary real numbers) form a subspace of $\mathbb{R}^4$.

15. Matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

16. Vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ form a basis of $\mathbb{R}^3$.

17. If the kernel of a matrix A consists of the zero vector only, then the column vectors of A must be linearly independent.

18. If the image of an $n \times n$ matrix A is all of $\mathbb{R}^n$, then A must be invertible.

19. If vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ span $\mathbb{R}^4$, then n must be equal to 4.

20. If vectors $\vec{u}, \vec{v}$, and $\vec{w}$ are in a subspace V of $\mathbb{R}^n$, then vector $2\vec{u}-3\vec{v}+4\vec{w}$ must be in V as well.

21. If A and B are invertible $n \times n$ matrices, then AB must be similar to BA.

22. If A is an invertible $n \times n$ matrix, then the kernels of A and A^{-1} must be equal.

23. Matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

24. Vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 9 \\ 8 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}$ are linearly independent.

25. If a subspace V of $\mathbb{R}^3$ contains the standard vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$, then V must be $\mathbb{R}^3$.

26. If a 2×2 matrix P represents the orthogonal projection onto a line in $\mathbb{R}^2$, then P must be similar to matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

27. If A and B are $n \times n$ matrices, and vector $\vec{v}$ is in the kernel of both A and B, then $\vec{v}$ must be in the kernel of matrix AB as well.

28. If two nonzero vectors are linearly dependent, then each of them is a scalar multiple of the other.

29. If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are any three distinct vectors in $\mathbb{R}^3$, then there must be a linear transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that $T(\vec{v}_1)=\vec{e}_1$, $T(\vec{v}_2)=\vec{e}_2$, and $T(\vec{v}_3)=\vec{e}_3$.

30. If vectors $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent, then vector $\vec{w}$ must be a linear combination of $\vec{u}$ and $\vec{v}$.

31. $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$.

32. If an $n \times n$ matrix A is similar to matrix B, then $A+7I_n$ must be similar to $B+7I_n$.

33. If V is any three-dimensional subspace of $\mathbb{R}^5$, then V has infinitely many bases.

34. Matrix I_n is similar to $2I_n$.

35. If $AB=0$ for two 2×2 matrices A and B, then BA must be the zero matrix as well.

36. If A and B are $n \times n$ matrices, and vector $\vec{v}$ is in the image of both A and B, then $\vec{v}$ must be in the image of matrix $A+B$ as well.

37. If V and W are subspaces of $\mathbb{R}^n$, then their union $V \cup W$ must be a subspace of $\mathbb{R}^n$ as well.

38. If the kernel of a 5×4 matrix A consists of the zero vector only and if $A\vec{v}=A\vec{w}$ for two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^4$, then vectors $\vec{v}$ and $\vec{w}$ must be equal.

39. If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ and $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ are two bases of $\mathbb{R}^n$, then there exists a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $T(\vec{v}_1)=\vec{w}_1$, $T(\vec{v}_2)=\vec{w}_2, \ldots, T(\vec{v}_n)=\vec{w}_n$.

40. If matrix A represents a rotation through $\pi/2$ and matrix B a rotation through $\pi/4$, then A is similar to B.

41. There exists a 2×2 matrix A such that $\text{im}(A)=\ker(A)$.

42. If two $n \times n$ matrices A and B have the same rank, then they must be similar.

43. If A is similar to B, and A is invertible, then B must be invertible as well.

44. If $A^2=0$ for a 10×10 matrix A, then the inequality $\text{rank}(A) \leq 5$ must hold.

45. For every subspace V of $\mathbb{R}^3$, there exists a 3×3 matrix A such that $V=\text{im}(A)$.

46. There exists a nonzero 2×2 matrix A that is similar to $2A$.

47. If the 2×2 matrix R represents the reflection about a line in $\mathbb{R}^2$, then R must be similar to matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.