MTH 201 – ASSIGNMENT 3

(1) A vector $v \in \mathbb{R}^n$ is called a *probability vector* if $v_i \geq 0$ and $\sum_{i=1}^n v_i = 1$. A square matrix $(n \times n)$ is called a *Markov matrix* if its columns are probability vectors.
a) Show that if A is Markov and v is a probability vector, then so is Av.
b) If A and B are $n \times n$ Markov matrices, then so is the product AB.

(2) The cross product of two vectors in $\mathbb{R}^3$ is given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix}.$$

Consider an arbitrary vector $v \in \mathbb{R}^3$. Is the transformation $T(x) = v \times x$ from $\mathbb{R}^3 \to \mathbb{R}^3$ linear? If so, find its matrix in terms of the components of the vector v.

(3) a) Let $u \in \mathbb{R}^3$ be a unit vector. Is the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by $T_u(x) = uu^t x$ linear? What does it do?
b) What does the transformation $I - 2T_u$ do? (where I denote the identity transformation $x \mapsto x$),
c) Let u, v and w be three mutually perpendicular unit vectors in $\mathbb{R}^3$. Describe what the transformation $T_u + T_v + T_w$ does.

(4) Use row reduction to find inverse of the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$.

(5) Use row reduction to find inverse of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$.

(6) Do the even numbered problems from 1-24 in the following list of T/F Questions. Supply reasons, not just T/F.

Chapter Two Exercises

TRUE OR FALSE?

1. The matrix $\begin{bmatrix} 5 & 6 \\ -6 & 5 \end{bmatrix}$ represents a rotation combined with a scaling.

2. If A is any invertible $n \times n$ matrix, then A commutes with A^{-1}.

3. The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ y - x \end{bmatrix}$ is a linear transformation.

4. Matrix $\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$ represents a rotation.

5. If A is any invertible $n \times n$ matrix, then $\mathrm{rref}(A) = I_n$.

6. The formula $(A^2)^{-1} = (A^{-1})^2$ holds for all invertible matrices A.

7. The formula $AB = BA$ holds for all $n \times n$ matrices A and B.

8. If $AB = I_n$ for two $n \times n$ matrices A and B, then A must be the inverse of B.

9. If A is a 3×4 matrix and B is a 4×5 matrix, then AB will be a 5×3 matrix.

10. The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1 \end{bmatrix}$ is a linear transformation.

11. Matrix $\begin{bmatrix} k & -2 \\ 5 & k-6 \end{bmatrix}$ is invertible for all real numbers k.

12. There exists a real number k such that the matrix $\begin{bmatrix} k-1 & -2 \\ -4 & k-3 \end{bmatrix}$ fails to be invertible.

13. There exists a real number k such that the matrix $\begin{bmatrix} k-2 & 3 \\ -3 & k-2 \end{bmatrix}$ fails to be invertible.

14. $A = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}$ is a regular transition matrix.

15. The formula $\det(2A) = 2\det(A)$ holds for all 2×2 matrices A.

16. There exists a matrix A such that
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

17. Matrix $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is invertible.

18. Matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is invertible.

19. There exists an upper triangular 2×2 matrix A such that $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

20. The function $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (y+1)^2 - (y-1)^2 \\ (x-3)^2 - (x+3)^2 \end{bmatrix}$ is a linear transformation.

21. There exists an invertible $n \times n$ matrix with two identical rows.

22. If $A^2 = I_n$, then matrix A must be invertible.

23. There exists a matrix A such that $A \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.

24. There exists a matrix A such that $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

25. The matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ represents a reflection about a line.

26. For every regular transition matrix A there exists a transition matrix B such that $AB = B$.

27. The matrix product $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is always a scalar multiple of I_2.

28. There exists a nonzero upper triangular 2×2 matrix A such that $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

29. There exists a positive integer n such that
$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^n = I_2.$$

30. There exists an invertible 2×2 matrix A such that $A^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

31. There exists a regular transition matrix A of size 3×3 such that $A^2 = A$.

32. If A is any transition matrix and B is any positive transition matrix, then AB must be a positive transition matrix.

33. If matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ is invertible, then matrix $\begin{bmatrix} a & b \\ d & e \end{bmatrix}$ must be invertible as well.

34. If A^2 is invertible, then matrix A itself must be invertible.

35. If $A^{17} = I_2$, then matrix A must be I_2.

36. If $A^2 = I_2$, then matrix A must be either I_2 or $-I_2$.

37. If matrix A is invertible, then matrix $5A$ must be invertible as well.

38. If A and B are two 4×3 matrices such that $A\vec{v} = B\vec{v}$ for all vectors $\vec{v}$ in $\mathbb{R}^3$, then matrices A and B must be equal.

39. If matrices A and B commute, then the formula $A^2B = BA^2$ must hold.

40. If $A^2 = A$ for an invertible $n \times n$ matrix A, then A must be I_n.

41. If A is any transition matrix such that A^{100} is positive, then A^{101} must be positive as well.

42. If a transition matrix A is invertible, then A^{-1} must be a transition matrix as well.

43. If matrices A and B are both invertible, then matrix $A + B$ must be invertible as well.

44. The equation $A^2 = A$ holds for all 2×2 matrices A representing a projection.

45. The equation $A^{-1} = A$ holds for all 2×2 matrices A representing a reflection.

46. The formula $(A\vec{v}) \cdot (A\vec{w}) = \vec{v} \cdot \vec{w}$ holds for all invertible 2×2 matrices A and for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$.

47. There exist a 2×3 matrix A and a 3×2 matrix B such that $AB = I_2$.

48. There exist a 3×2 matrix A and a 2×3 matrix B such that $AB = I_3$.

49. If $A^2 + 3A + 4I_3 = 0$ for a 3×3 matrix A, then A must be invertible.

50. If A is an $n \times n$ matrix such that $A^2 = 0$, then matrix $I_n + A$ must be invertible.