MTH102 week2 Jan 8,9,11

02 January 2019 11:40

Recap from previous week

Observation: If $\vec{u}, \vec{v}$ are unit vectors

then $-1 \leq \vec{u}\cdot\vec{v} \leq 1$

Proof: Let $t \in \mathbb{R}$, since $\|u + t\vec{v}\| \geq 0$

Squaring we get $(\vec{u}+t\vec{v})\cdot(\vec{u}+t\vec{v}) \geq 0$

$\|u\|^2 + t^2\|v\|^2 + 2t\,\vec{u}\cdot\vec{v} \geq 0$

$1 + t^2 + 2t\,\vec{u}\cdot\vec{v} \geq 0$

If $t=1$ we get $\vec{u}\cdot\vec{v} \geq -1$

If $t=-1$ we get $\vec{u}\cdot\vec{v} \leq 1$ QED

Thm Cauchy-Schwarz inequality

$|\vec{u}\cdot\vec{v}| \leq \|u\|\cdot\|v\|$

Pf: If $\vec{u}$ or $\vec{v} = 0$, proof is clear.

Otherwise $-1 \leq \frac{\vec{u}}{\|\vec{u}\|}\cdot\frac{\vec{v}}{\|v\|} \leq 1$ by

observation above. QED.

Corollary: Δ-inequality $\|u+v\| \leq \|u\| + \|v\|$

Pf: $\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2\,\vec{u}\cdot\vec{v}$

by Cauchy Schwarz $2\,\vec{u}\cdot\vec{v} \leq 2\|u\|\|v\|$

So $\|u+v\|^2 \leq (\|u\| + \|v\|)^2$. QED

Def^n: Projection of $\vec{v}$ on $\vec{u}$ is

$$Pr_{\vec{u}}\vec{v} = (\vec{v}\cdot\hat{u})\,\hat{u} \quad \text{where}$$

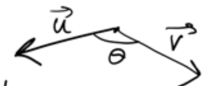
$$\hat{u} = \vec{u}/\|\vec{u}\|.$$

Property: $\vec{v} - Pr_{\vec{u}}\vec{v}$ is $\perp$ to $\vec{u}$

Def^n: Angle between two non-zero vectors $\vec{u}$ and $\vec{v}$ is

$$\cos^{-1}(\hat{u}\cdot\hat{v}) = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|u\|\,\|v\|}\right)$$

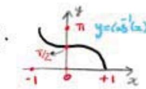
We always consider the smaller of the 2 possible angles between a pair of vectors



angles between a pair of vectors

This angle is between 0 and π

The range of $\cos^{-1}$ is also $[0,\pi]$.



CROSS-PRODUCT in $\mathbb{R}^3$

We start with determinant of 2x2, 3x3 matrices

- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$
- $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$ — (1)

If we expand along i-th row instead of first row, we have to multiply by $(-1)^{i-1}$.

first row, we have to multiply by (-1)

So the same determinant is also

$$-\left(b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}\right) \quad \text{and}$$

$$+\left(c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\right) \quad — (2)$$

Example: $\begin{vmatrix} 1 & 3 & 5 \\ 0 & 2 & 7 \\ -1 & 0 & 3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & 7 \\ 0 & 3 \end{vmatrix} - 3 \begin{vmatrix} 0 & 7 \\ -1 & 3 \end{vmatrix} + 5 \begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix}$

$= 6 - 21 + 10 = -5$

$= -0 \cdot \begin{vmatrix} 3 & 5 \\ 0 & 3 \end{vmatrix} + 2 \begin{vmatrix} 1 & 5 \\ -1 & 3 \end{vmatrix} - 7 \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} = 2 \cdot 8 - 21 = -5$

- If we interchange two rows, then det acquires a negative sign — (3)

Pf: Use (1) and (2)

- $\begin{vmatrix} \lambda u^t \\ v^t \\ \cdot \end{vmatrix} = \begin{vmatrix} u^t \\ \lambda v^t \\ \cdot \end{vmatrix} = \begin{vmatrix} u^t \\ v^t \\ \lambda w^t \end{vmatrix} = \lambda \begin{vmatrix} u^t \\ v^t \\ w^t \end{vmatrix}$ — (4)

- $\begin{vmatrix} \lambda u^t \\ v^t \\ w^t \end{vmatrix} = \begin{vmatrix} u \\ \lambda v^t \\ w^t \end{vmatrix} = \begin{vmatrix} \cdot \\ v^t \\ \lambda w^t \end{vmatrix} = \lambda \begin{vmatrix} \cdot \\ v^t \\ w^t \end{vmatrix}$ — (1)

- $\begin{vmatrix} a & b \\ c+c' & d+d' \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ c' & d' \end{vmatrix}$ — (5)

Pf: $a(d+d') - b(c+c') = ad - bc + ad' - bc'$

- $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1+c_1' & c_2+c_2' & c_3+c_3' \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1' & c_2' & c_3' \end{vmatrix}$ — (6)

Pf of (6): Use (5) in (1)

Using (6) and (3):

$$\begin{vmatrix} u^t \\ v^t + \tilde{v}^t \\ w^t \end{vmatrix} = \begin{vmatrix} u^t \\ v^t \\ w^t \end{vmatrix} + \begin{vmatrix} u^t \\ \tilde{v}^t \\ w^t \end{vmatrix}$$

and from (1) itself.

and from (1) itself:

$$\begin{vmatrix} u^t + \tilde{u}^t \\ v^t \\ w^t \end{vmatrix} = \begin{vmatrix} u^t \\ v^t \\ w^t \end{vmatrix} + \begin{vmatrix} \tilde{u}^t \\ v^t \\ w^t \end{vmatrix}$$

- If $u \parallel v$ in $\mathbb{R}^2$ then $\begin{vmatrix} u^t \\ v^t \end{vmatrix} = 0$

Pf $\begin{vmatrix} a & b \\ \lambda a & \lambda b \end{vmatrix} = \lambda ab - \lambda ab = 0$

Using this in (1) and (2), we get:

If $u, v, w \in \mathbb{R}^3$ with $u \parallel v$ or $u \parallel w$ or $v \parallel w$

then $\begin{vmatrix} u^t \\ v^t \\ w^t \end{vmatrix} = 0 \quad \longrightarrow (7)$

Given $\vec{u}, \vec{v}$ in $\mathbb{R}^3$, we define another vector in $\mathbb{R}^3$ called $\vec{u} \times \vec{v}$. In a seperate (optional) notes, we will discuss why we define cross product only in $\mathbb{R}^3$ and what can be done in $\mathbb{R}^n$.

<u>Def^n</u>: $\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1)^t$

Symbolically $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

<u>Example</u>: $(3\hat{i} - 2\hat{j} + \hat{k}) \times (\hat{i} + \hat{j} + \hat{k}) =$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \hat{i}(-3) - \hat{j}(2) + \hat{k}(5)$$

Properties of $\vec{u} \times \vec{v}$

1) If $v \parallel u$ then $\vec{u} \times \vec{v} = 0$

2) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

3) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

and $\vec{u} \times (c\vec{v}) = c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v}$

4) $\vec{u} \times \vec{v}$ is $\perp$ to both $\vec{u}, \vec{v}$

If $\vec{u} \nparallel \vec{v}$ then $\vec{u} \times \vec{v}$ is $\perp$ to plane spanned by $\vec{u}$ and $\vec{v}$

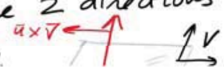
Pf: $\vec{w} \cdot (\vec{u} \times \vec{v}) = \det \begin{pmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}$

$= \vec{u} \cdot (\vec{v} \times \vec{w})$ (**Scalar Triple Product of w,u,v**)

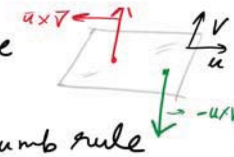
$= \vec{v} \cdot (\vec{w} \times \vec{u})$

It is now clear that $\vec{u} \cdot (\vec{u} \times \vec{v}) = \vec{v} \cdot (\vec{u} \times \vec{v}) = 0$

Suppose $u \nparallel v$, there are 2 directions



for normal to u-v plane



The direction of $\vec{u} \times \vec{v}$ is given by right-hand thumb rule

"CURL fingers of right hand from u to v then thumb points along $\vec{u} \times \vec{v}$

5) $\|u \times v\| = \|u\| \, \|v\| \sin\theta$

where θ is smaller angle btw u, v

Pf: We expand $\|u \times v\|^2$ as

$(u_2v_3)^2 + (u_3v_2)^2 + (u_1v_3)^2 + (u_3v_1)^2 + (u_1v_2)^2 + (u_2v_1)^2$

$-2u_2v_2u_3v_3 - 2u_1v_1u_3v_3 - 2u_1v_1u_2v_2$

$= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2$

$= \|u\|^2 \|v\|^2 (1 - \cos^2\theta)$

$= (\|u\| \, \|v\| \sin\theta)^2$

$= \quad (\|u\| \, \|v\| \sin\theta)^2$

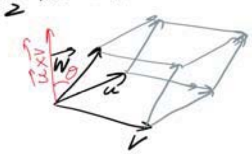
Since $\sin\theta \geq 0$ for $0 \leq \theta \leq \pi$ we set

$\|u \times v\| = \|u\| \, \|v\| \sin\theta$

COROLLARY : Area of parallelogram

is $\|\vec{u} \times \vec{v}\|$ and Δ is $\frac{1}{2}\|\vec{u} \times \vec{v}\|$

Volume of box is Area of base $\times$ height $=$

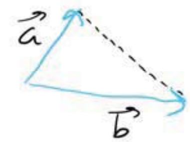


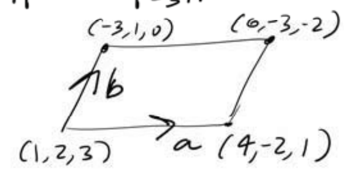
$\|\vec{u} \times \vec{v}\| \, \|\vec{w}\| \cos\theta = |\vec{w} \cdot (\vec{u} \times \vec{v})|$

Examples • Area of Δ determined by $\vec{a} = \hat{i} + \hat{j}$ and $\vec{b} = 2\hat{i} - \hat{j}$

$= \frac{1}{2} \, \| \, | \hat{i} \quad \hat{j} \quad \hat{k} | \, \| = \frac{1}{2} \left\| \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\| = \frac{3}{2}$

$= \frac{1}{2} \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 2 & -1 & 0 \end{vmatrix} \right\| = \frac{1}{2} \left\| \begin{pmatrix} 0 \\ 0 \\ -3 \end{pmatrix} \right\| = \frac{3}{2}$





• Area of Parallelogram with vertices $(1,2,3)$, $(4,-2,1)$, $(-3,1,0)$, $(0,-3,-2)$

$\vec{a} = \begin{pmatrix} 3 \\ -4 \\ -2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -4 \\ -1 \\ -3 \end{pmatrix}$; Area $= \|a \times b\|$

$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & -2 \\ -4 & -1 & -3 \end{vmatrix} = \hat{i}(10) - \hat{j}(-17) + \hat{k}(-19)$

$\|a \times b\| = \sqrt{10^2 + 17^2 + 19^2} = 5\sqrt{30}$

• Area of Parallelepiped spanned by $\vec{u}, \vec{v}, \vec{w}$ given by $\begin{pmatrix} -1 \\ -2 \end{pmatrix} \begin{pmatrix} 4 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

given by $\begin{pmatrix}-1\\-2\\0\end{pmatrix}, \begin{pmatrix}4\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\6\end{pmatrix}$.

Ans: Abs value of $\begin{vmatrix}0 & 1 & 6\\ -1 & -2 & 0\\ 4 & -1 & -1\end{vmatrix} = -1(1)+6(9) = 53$

<u>Exercise</u>: 1) Let $\hat{u}$ be a unit vector and $\vec{v} \perp \hat{u}$ in $\mathbb{R}^3$. Show $\hat{u}\times(\hat{u}\times\vec{v}) = -\vec{v}$

<u>Pf</u>: Let $w = \hat{u}\times v$. Since w is $\perp$ to u-v plane, it follows $\hat{u}\times w$ is in u-v plane and $\perp$ to $\hat{u}$, hence $\hat{u}\times w = cv$ for some $c \in \mathbb{R}$. Now

$c\|v\|^2 = \vec{v}\cdot\hat{u}\times w = w\cdot\vec{v}\times\hat{u}$

$= -(\hat{u}\times v)\cdot(\hat{u}\times v) = -\|\hat{u}\times v\|^2$

$= -\|v\|^2 \Rightarrow c = -1$. QED

2) $\hat{u}\in\mathbb{R}^3, v\in\mathbb{R}^3$, then

$\hat{u}\times(\hat{u}\times\vec{v}) = -(v - Pr_u v)$

<u>Pf</u>: $\hat{u}\times(\hat{u}\times\vec{v}) = \hat{u}\times(\hat{u}\times(v - Pr_u v))$

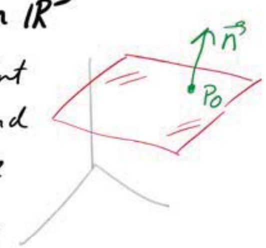
(by previous part) $= -(v - Pr_u v)$

Lastly $u, v \in \mathbb{R}^3$ then

$u\times(u\times v) = -\|u\|^2(v - Pr_u v)$ — (8)

Equation of plane in $\mathbb{R}^3$

let $(x_0, y_0, z_0) = P_0$ be a point lying on the plane and $\vec{n} = \begin{pmatrix}a\\b\\c\end{pmatrix}$ is $\perp$ to plane

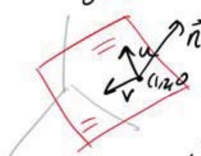


i.e. $P = (x ...)$ generic pt on

Let $P = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ generic pt on the plane. Then clearly $\begin{pmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{pmatrix} \perp \vec{n}$

Thus $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ is eq of plane.

<u>Example</u>: Find eq^n of plane in $\mathbb{R}^3$ passing through $(1,2,0), (3,1,2), (0,1,1)$



$\vec{u} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

$\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 2 \\ -1 & -1 & 1 \end{vmatrix} = \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix}$

$1(x-1) + (-4)(y-2) + -3(z-0) = 0$

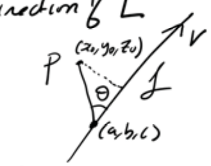
$x - 4y - 3z = -7$

$x - 4y - 3z = -7$

- Distance of point from line

Let L be a line in $\mathbb{R}^n$, P a point in $\mathbb{R}^n$ not lying on L.

Let $(a,b,c) \in \mathcal{L}$, $\vec{v}$ direction of L and $P = (x_0, y_0, z_0)$

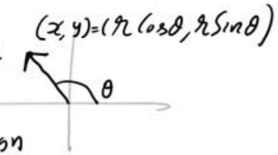


$\mathrm{DIST}(P, \mathcal{L})$ is

$\left\| \begin{matrix} x_0-a \\ y_0-b \\ z_0-c \end{matrix} \right\| \sin\theta = \left\| \begin{pmatrix} x_0-a \\ y_0-b \\ z_0-c \end{pmatrix} \times \hat{v} \right\|$

where $\hat{v} = v/\|v\|$

- Polar coords in $\mathbb{R}^2$

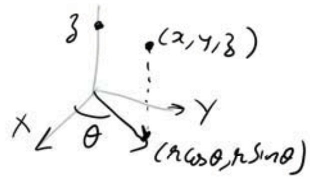


r distance from origin

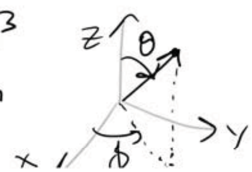
r distance from origin

θ measured c.c.w from the x-axis

- Cylindrical coords in $\mathbb{R}^3$
  (r, θ, z) where (r, θ) = polar coords of projection to x-y plane



- Spherical coords in $\mathbb{R}^3$



ρ = distance from origin

$\theta \in [0, \pi]$ elevation from the z-axis

ϕ azimuth : polar angle coordinate of projection to x-y plane

$$x = \rho \sin\theta \cos\phi$$
$$y = \rho \sin\theta \sin\phi$$
$$z = \rho \cos\theta$$

HW2

Q1) Show $\begin{vmatrix} (u+\lambda v)^t \\ v^t \\ w^t \end{vmatrix} = \begin{vmatrix} u^t \\ v^t \\ w^t \end{vmatrix} \quad \forall \lambda \in \mathbb{R}$

Adding multiple of Row i to Row j, $i \neq j$ does not affect a 3×3 determinant "does not affect" a 3×3 determinant

Q2) Area of Δ with vertices $(1,0,1), (0,2,3), (-1,5,-2)$

Q3) Volume of Parallelepiped with vertices
$(3,0,-1), (4,2,-1), (-1,1,0), (3,1,5), (0,3,0),$
$(4,3,5), (-1,2,6), (0,4,6)$ (TRICKY)

$(\vec{u},\vec{v},\vec{w}) = \begin{pmatrix}-1\\-2\\0\end{pmatrix}, \begin{pmatrix}4\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\6\end{pmatrix}$ $\to$ Vol $= 53$ by inclass example

Q4) let $f: \mathbb{R}^3 \to \mathbb{R}$ be a function
satisfying i) $f(\bar{x}+\bar{y}) = f(x)+f(y)$, ii) $f(c\vec{x}) = cf(x)$
Show there is a unique $\vec{u} \in \mathbb{R}^3$ such that
$f(x) = \vec{u}\cdot\vec{x}$. Is this true in $\mathbb{R}^n$?

optional

Q5) * The "$\mathbf{ca\bar{b} - ba\bar{c}}$" rule: (optional)

$$a\times(b\times c) = \vec{b}(\bar{a}\cdot\bar{c}) - \vec{c}(\bar{a}\cdot\bar{b}) \qquad (9)$$

Fill details of the Proof outline below

1) Show $a\times(b\times c)$ lies in b-c plane
2) First assume $\vec{c}$ is $\perp$ to $\vec{b}$
3) Fix $\vec{c}, \vec{b}$ with $\vec{c}\perp\vec{b}$ and $\vec{a}$ is Variable
By Q4) and step 1)
$$\bar{a}\times(\bar{b}\times\bar{c}) = \vec{c}(\vec{a}\cdot\vec{u}) + \vec{b}(\vec{a}\cdot\vec{v})$$
for some $\vec{u}, \vec{v}$ (unique) in $\mathbb{R}^3$
4) $\vec{u}, \vec{v} \in b$-c plane (Use $\bar{a} = \bar{b}\times\vec{c}$)
5) Set $\bar{a} = \vec{b}$ and $\vec{a} = \vec{c}$ and use (8) and $\vec{c}\perp\vec{b}$ to conclude

use (b) and ...

$\vec{u} = -\vec{b}$ and $\vec{v} = \vec{c}$

6) Now relax the condition $\vec{c} \perp \vec{b}$

(Resolve $c = pr_b c + c - pr_b c$)

Q6) $(a \times b) \cdot (c \times d) = (\bar{a} \cdot \bar{c})(\bar{b} \cdot \bar{d}) - (\bar{b} \cdot \bar{c})(\bar{a} \cdot \bar{d})$

(Use properties of "Scalar Triple product" and the result of Q5)

Q7) Two planes with normals n_1, n_2 are parallel $\Longleftrightarrow n_2 \parallel n_1$

If $n_1 \not\parallel n_2$ then the planes intersect in a unique line. If (x_0, y_0, z_0) is on this line then, prove that param. eqⁿ of line is

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\, \vec{n_1} \times \vec{n_2}$$

Find parametric eqⁿ of line which is intersection of planes $x + y = 1$, $y + z = 1$

Find acute angle of intersection of the planes

In general give formula for acute angle of intersection in terms of n_1, n_2.

Q8) Find distance of $(-11, 10, 20)$ from the line $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 5-t \\ 3 \\ 8+7t \end{pmatrix}$