Indian Institute of Science Education and Research, Pune

END SEMESTER EXAMINATION, JANUARY 2019

Course name: Multivariable Calculus **Course code:** MTH 102
Date: April 24, 2019 *Duration:* 2 hours
Instructor: Krishna Kaipa Total points: 35

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1. (a) (5 points) The cylinder $x^2 + y^2 = 4$ and the plane $2x + 2y + z = 2$ intersect in an ellipse. Find the points on the ellipse which are farthest from the origin. Hint: Express the objective functionas a function of one variable θ.
1 Bonus points: Find the points on the ellipse which are nearest from the origin.

Solution: The ellipse can be parametrized as $\theta \mapsto (2\cos\theta, 2\sin\theta, 2 - 4\cos\theta - 4\sin\theta)$ for $0 \leq \theta < 2\pi$. The distance squared from the origin of the point with parameter θ is thus $f(\theta) = 4 + (2 - 4\cos\theta - 4\sin\theta)^2$ **(1.5 Points for correct objective function)**. Effectively, we have to find the largest value of $(\cos\theta + \sin\theta - 1/2)^2$ or $(\sin(\theta + \pi/4) - 1/\sqrt{8})^2$. The largest value is clearly obtained when $\sin(\theta + \pi/4) = -1$, which has the unique solution $\theta = 5\pi/4$ **(2.5 Points for correct θ)**. The desired point is $(-\sqrt{2}, -\sqrt{2}, 2 + 4\sqrt{2})$ **(1 Point for correct final answer)**.
The nearest points are clearly those for which $(\sin(\theta + \pi/4) - 1/\sqrt{8}) = 0$. This occurs for $\theta = -\pi/4 + \sin^{-1}(1/\sqrt{8})$ and $\theta = 3\pi/4 - \sin^{-1}(1/\sqrt{8})$.
OR
Lagrange Multiplier Method:
Writing objective function and all 5 equations correctly **(1 Point for correct equations)**
Discussed x=y case **(1 Point)**
Discussed $\lambda_1 = 1$ case **(1 Point)**
Found all 4 critical points **(1 Point)**
Correct final answer **(1 Point)**

(b) (4 points) A dome is shaped as a hemisphere of radius a. If a pole whose length is the average height of the dome is to be installed inside the dome in a vertical position, where on the floor can it be located?

Solution: The average height is $(\iint z dA)/(2\pi a^2)$. Writing $z = f(x, y) = \sqrt{a^2 - x^2 - y^2}$ we have $dA = \sqrt{1 + f_x^2 + f_y^2} dxdy = adxdy/\sqrt{a^2 - x^2 - y^2} = ardrd\theta/(\sqrt{a^2 - r^2})$. So the average height is

$$\frac{2\pi a}{2\pi a^2} \int_{r=0}^{a} rdr\sqrt{a^2 - r^2}/\sqrt{a^2 - r^2} = a/2 \textbf{(3 Points for correct value)}$$

So the pole must be placed at a distance of $\sqrt{3}a/2$ from the origin **(1 Point for correct final answer)**.
Remark:
Some students have done it using integration on base (circle). They have been given 2 marks.

2. (a) (4 points) Show that 2 is the average value of $f(x, y, z) = z^2 + xe^y$ on the curve C obtained by intersecting the (elliptic) cylinder $x^2/5 + y^2 = 1$ by the plane $z = 2y$.

Solution: The curve is parametrized as $t \mapsto (\sqrt{5}\cos t, \sin t, 2\sin t)$ for $0 \leq t \leq 2\pi$ **(1 Point for correct parameterization)**. The speed is $\sqrt{5}$. So the average value **(1 Point for correct formula for average)** of f is

$$\frac{1}{2\pi\sqrt{5}} \int_0^{2\pi} (4\sin^2 t + \sqrt{5}\cos t\, e^{\sin t})\sqrt{5}\, dt =$$

Since $\int_0^{2\pi} 4\sin^2 t dt = 4\pi$ and $\int_0^{2\pi} e^{\sin t} d\sin t = 0$, the average value is $\frac{4\pi\sqrt{5}}{2\pi\sqrt{5}} = 2$ **(1.5 Point for correct value of integral of function over C) (1 Point for correct length of curve)**.

(b) (4 points) The stem of a mushroom is modeled as a right circular cylinder with radius 1/2, height 2, and its cap is modeled as a hemisphere of radius R. If the mushroom has axial symmetry, is of uniform density, and its center of mass lies at the center of plane where the stem joins the cap, then find R.

Solution: The center of gravity of a hemisphere of radius R is

$$\frac{1}{2\pi R^3/3} \int_{\theta=0}^{2\pi} \int_{r=0}^{R} \int_{z=0}^{\sqrt{R^2-r^2}} zdzrdrd\theta = 3R/8.\textbf{(1.5 Point)}$$

(1 Point for correct computation of center of gravity of cylinder) Using this the height of the center of gravity of the mushroom is:

$$2 = \frac{(2\pi R^3/3)\cdot(2+3R/8)+(\pi\cdot(1/2)^2\cdot 2)\cdot 1}{2\pi R^3/3+\pi\cdot(1/2)^2\cdot 2}$$

This gives $R = 2^{1/4}$ **(1.5 Point for center of gravity of mushroom)**.

3. (a) (5 points) Evaluate the line integral $\oint_C(-x^2y + e^{x^2})dx + (xy^2 + \sin(y^2))dy$ where C is the boundary of the region in the first quadrant bounded by the hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 9$, $y = 2/x$ and $y = 4/x$ traced once in the counterclockwise direction.

Solution: By Green's theorem **(2 Point Using Green's Theorem)** the integral is equal to $\iint(y^2+x^2)dxdy$ over the region bounded by the 4 hyperbolas. Let $u = x^2 - y^2$ and $v = xy$. Let $R = [1,9] \times [2,4]$ in uv-plane. By change of variable **(2 Point for change of transformation with Jacobian (0.5 point is deducted if Jacobian is wrong))** the above double integral equals

$$\iint_R (x^2+y^2)dudv/(2(x^2+y^2)) = (9-1)(4-2)/2 = 8$$

(1 Point for final calculations with limits)

(b) (4 points) Compute the surface area of the part of the paraboloid $x^2 + z^2 = 3ay$ for which $0 \leq y \leq a$.

Solution: (1.5 Point for correct parameterization of the surface) The surface is given as the graph $y = f(x,z) = (x^2+z^2)/3a$. The area element is

$$dA = dxdz\sqrt{1 + {f_x}^2 + {f_z}^2} = rdrd\theta\sqrt{1 + 4r^2/9a^2} = 9a^2/8\sqrt{t}d\theta dt$$

where $t = 1 + 4r^2/9a^2$ **(1 Point for calculating $R_u \times R_v$)**. The limits on r are from 0 to $a\sqrt{3}$, and hence the area is

$$\int_{\theta=0}^{2\pi}\int_{t=1}^{7/3} 9a^2/8\sqrt{t}d\theta dt = \tfrac{3\pi a^2}{2}\left((7/3)^{3/2} - 1\right)$$

(1.5 Point for correct evaluation of surface integral)

4. (a) (5 points) Evaluate the line integral $\oint_C (y^2-z^2)dx + (z^2-x^2)dy + (x^2-y^2)dz$ where C is the curve cut out of the cube $[0,a] \times [0,a] \times [0,a]$ by the plane $x+y+z = 3a/2$. Orient C such that its projection on xy-plane is oriented clockwise.

Solution: Let the vertor field is $F(x,y,z) = (y^2-z^2)\mathbf{i} + (z^2-x^2)\mathbf{j} + (x^2-y^2)\mathbf{k}$.
The curl of the vector field is $(\nabla \times F) = -2((y+z),(x+z),(x+y))$. $\left[\bullet 1 \text{ point}\right]$
The unit normal vector is $(1,1,1)/\sqrt{3}$. $\left[\bullet \text{ 0.5 point }\right]$
Now given orientation is clockwise so the desired unit normal vector will be $\hat{n} = -(1,1,1)/\sqrt{3}$. $\left[\bullet \text{ 0.5 point }\right]$
We will use Stokes' Theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\vec{\nabla} \times \vec{F}) \cdot \hat{n}dA$. $\left[\bullet \text{ 0.5 point }\right]$
The normal component of curl is thus $4(x+y+z)/\sqrt{3}$. In the region R bounded by C we have $x+y+z = 3a/2$ and hence the line integral will be $\oint_C \vec{F} \cdot d\vec{r} = \iint_R \frac{4(x+y+z)}{\sqrt{3}}dA = \iint_R \frac{4}{\sqrt{3}} \cdot \frac{3a}{2}dA = 2\sqrt{3}a \cdot \text{Area}(R)$ OR $6a \cdot$ Area of projected region. $\left[\bullet \text{ 1 point }\right]$

The region R is a regular hexagon of side length $a/\sqrt{2}$ OR the projected region is hexagon with length of four sides is $a/2$ and other two have $a/\sqrt{2}$. [$\bullet$ 1 point]
and hence its area is $3\sqrt{3}a^2/4$ OR area of projected region is $\frac{3a^2}{4}$ [$\bullet$ 0.5 point]
Thus the answer is $2\sqrt{3}a \cdot 3\sqrt{3}a^2/4$ OR $6a \cdot \frac{3a^2}{4} = 9a^3/2$.

(b) (4 points) Let W be the three-dimensional solid enclosed by the surfaces $x = y^2$, $x = 9$, $z = 0$, and $x = z$. Let S be the boundary of W. Find the flux $\iint \vec{F} \cdot d\vec{S}$ of $F(x, y, z) = (3x - 5y)\mathbf{i} + (4z - 2y)\mathbf{j} + 8yz\mathbf{k}$ across S.

Solution:

We will use Gauss's Divergence Theorem $\iint_S \vec{F} \cdot d\vec{S} = \iiint_V \operatorname{div}(\vec{F})dV$. [$\bullet$ 0.5 point]
The divergence i.e $\operatorname{div}(\vec{F}) = 1 + 8y$. [$\bullet$ 1 point]
By divergence theorem the flux equals

$$\int_{y=-3}^{3} \int_{x=y^2}^{9} \int_{z=0}^{x} (1 + 8y)\, dz dx dy \quad [\bullet\ 1.5\text{point}]$$

$$= \int_{y=-3}^{3} (\frac{1}{2} + 4y)(81 - y^4)dy = \int_{-3}^{3} (\frac{81}{2} + 324y - \frac{y^4}{2} - 4y^5)dy \quad [\bullet\ 0.5\text{point}]$$

$$= 2\int_{0}^{3} (\frac{81}{2} - \frac{y^4}{2})dy = \int_{0}^{3} (3^4 - y^4)dy \quad [\bullet\ 0.5\text{point}]$$

$= 3^5 \cdot \frac{4}{5} = \frac{972}{5}$