

# The discrete logarithm problem in the group of non-singular circulant matrices

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**Abstract.** The discrete logarithm problem is one of the backbones in public key cryptography. In this paper we study the discrete logarithm problem in the group of circulant matrices over a finite field.

**Keywords.** The discrete logarithm problem, circulant matrices.

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## 1 Introduction

Menezes and Wu [5] claim that working with the discrete logarithm problem in matrices offers no major improvement from working with a finite field. Many authors, including myself [3], repeated that claim. It is now a common knowledge that for practical purposes, the discrete logarithm problem in non-singular matrices is not worth looking at.

In this note, I provide a counterexample to the above mentioned common knowledge and show that matrices over a finite field can be used effectively to produce a fast and secure cryptosystem. This approach can be seen as working with the MOR cryptosystem [4], with finite dimensional vector spaces over a finite field.

In this note, we are dealing with the discrete logarithm problem in matrices over a finite field, i.e., given a non-singular  $d \times d$  matrix  $A$  and  $B = A^m$  over  $\mathbb{F}_q$ , compute the *private key*  $m$ ; where  $q$  is a power of a prime  $p$ . One can easily build any cryptosystem that uses the discrete logarithm problem, like the Diffie–Hellman key exchange or the ElGamal cryptosystem, using the discrete logarithm problem in matrices. There are many aspects to the security of a cryptosystem. In this paper we will only deal with the computational aspects of solving a discrete logarithm problem.

The core of the Menezes–Wu algorithm [5, Algorithm 2] is to compute the characteristic polynomial  $\chi_A(x)$  of  $A$ . The eigenvalues of  $A$ , which are the roots of  $\chi_A(x)$  belong to the splitting field of  $\chi_A(x)$ . The roots of  $\chi_B(x)$  also belong to the same splitting field. Then to solve the discrete logarithm problem, one has to

solve the individual discrete logarithm problems in the eigenvalues and then use the Chinese remainder theorem. The security of the discrete logarithm problem depends on the degree of the extension of the splitting field. Since solving a discrete logarithm problem depends on the size of the field, we can get excellent security by taking  $d$  large (around 20) and choose  $A$  such that  $\chi_A(x)$  is irreducible. However, in that case matrix multiplication becomes very expensive and we are better off working with the finite field  $\mathbb{F}_{q^d}$ . This is the argument of Menezes and Wu [5].

In this paper, we deal with a particular type of non-singular matrices – the *circulant matrices*. We show, that for these matrices, squaring is free and multiplication is easy. When this is the case, the above argument is no longer valid and we have possibilities of a successful cryptosystem – that is secure with reasonable parameters and is fast. Using the extended Euclidean algorithm, computing the inverse of a circulant matrix is easy, that makes a cryptosystem built on circulant matrices very fast and secure.

When working with the discrete logarithm problem in matrices, one should be careful of the fact that the determinant of a matrix is a multiplicative function to the ground field. This can always reduce the discrete logarithm problem in matrices to a discrete logarithm problem in the ground field. This can be easily avoided by:

- (i) Choose  $A$  such that determinant of  $A$  is 1.

In this note, the Roman numerals (i), (ii),  $\dots$ , (v), are the set of conditions on the circulant matrix  $A$ . All matrices are over a finite field.

## 2 Circulant matrices

The reader is reminded that all fields (often denoted by  $F$ ) are finite with characteristic  $p$ .

**Definition 2.1.** A  $d \times d$  matrix over a field  $F$  is called circulant, if every row except the first row, is a right circular shift of the row above that. So a circulant matrix is defined by its first row. One can define a circulant matrix similarly using columns.

Even though a circulant matrix is a two dimensional object, in practice it behaves much like a one dimensional object; given by the first row or the first column. We will denote a circulant matrix  $C$  with first row  $c_0, c_1, \dots, c_{d-1}$  by

$C = \text{circ}(c_0, c_1, c_2, \dots, c_{d-1})$ . An example of a circulant  $5 \times 5$  matrix is:

$$\begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_4 \\ c_4 & c_0 & c_1 & c_2 & c_3 \\ c_3 & c_4 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_4 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_4 & c_0 \end{pmatrix}$$

It is easy to see that all the (sub)diagonals of a circulant matrix are constant. This fact comes in handy. Let  $W = \text{circ}(0, 1, 0, \dots, 0)$  be a  $d \times d$  circulant matrix, then clearly  $W^d = I$ . We can write  $C = c_0I + c_1W + c_2W^2 + \dots + c_{d-1}W^{d-1}$ . One can define a *representer polynomial* corresponding to the circulant matrix  $C$  as  $\phi_C = c_0 + c_1x + c_2x^2 + \dots + c_{d-1}x^{d-1}$ . This shows that the circulants form a commutative ring with respect to matrix multiplication and matrix addition and is isomorphic to (the isomorphism being matrix to representer polynomial)  $\frac{F[x]}{x^d-1}$ . For more on circulant matrices, see [1].

### 2.1 How easy is it to square a circulant matrix?

Let  $A = \text{circ}(a_0, a_1, \dots, a_{d-1})$  be a circulant matrix over a field of characteristic 2. We show that to compute  $A^2$ , we need to compute  $a_i^2$  for each  $i$  in  $\{0, 1, 2, \dots, d-1\}$ . Then  $A^2 = \text{circ}(a_{\pi(0)}^2, a_{\pi(1)}^2, \dots, a_{\pi(d-1)}^2)$ , where  $\pi$  is a permutation of  $0, 1, 2, \dots, (d-1)$ . This was also observed by Silverman [8, p. 180].

**Theorem 2.2.** *If the characteristic of the field  $F$  is 2, and  $d$  is an odd integer, then squaring a  $d \times d$  circulant matrix  $A$  is the same as squaring  $d$  field elements.*

*Proof.* We use the standard method of matrix multiplication; where one computes the dot product of the  $i^{\text{th}}$  row with the  $j^{\text{th}}$  column for the element at the intersection of the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column of the product matrix. As we saw before the circulant matrices are closed under multiplication and a circulant matrix is given by its first row.

Taking these into account, if the circulant matrix is  $A = \text{circ}(a_0, a_1, \dots, a_{d-1})$ , we see that the first element of the first row of the product, is the dot product of  $(a_0, a_1, \dots, a_{d-1})$  with the first column  $(a_0, a_{d-1}, \dots, a_1)^T$ . The first column can be thought of as the map  $a_i \mapsto a_{-i \pmod d}$  for  $i = 0, 1, \dots, (d-1)$ .

For each  $j$  in  $\{0, 1, 2, \dots, d-1\}$ , the map is given by  $a_i \mapsto a_{j-i \pmod d}$ . Now notice that if  $i \mapsto j - i \pmod d$ , then  $j - i \mapsto i \pmod d$ . This proves that there are pairs formed in the dot product, which makes it zero when working in characteristic 2.

The only thing that escapes forming pairs, are those  $i$ , for which  $i = j - i \pmod{d}$ . Since  $d$  is odd, there is an inverse of  $2 \pmod{d}$  and an unique solution for  $i$ .  $\square$

It is easy to see from the above proof, that once an odd integer  $d$  is fixed, one can easily compute the permutation  $\pi$ . The computation of  $d$  different powers can be done in parallel.

### 3 The discrete logarithm problem in the group of non-singular circulant matrices

As we saw before, circulant matrices can be represented in two different ways – one as a circulant matrix and other as an element of the ring  $\mathcal{R} = \frac{F[x]}{x^d-1}$ . In the later case, each element of  $\mathcal{R}$  is a polynomial of degree  $d-1$  in  $F$ . The polynomial multiplication in  $\mathcal{R}$  can be done (in parallel) using matrix multiplication. If matrix multiplication is used to do the polynomial multiplication, then there is no need to do the reduction mod  $x^d - 1$ .

These two representations lead to two different kinds of attack to the discrete logarithm problem:

- (a) The discrete logarithm problem in matrices.
- (b) The discrete logarithm problem in  $\mathcal{R}$ .

#### 3.1 The discrete logarithm problem in matrices

As we understood from Menezes and Wu [5], solving the discrete logarithm problem in non-singular matrices is tied to the largest degree of the irreducible component of the characteristic polynomial. The best case scenario happens when the characteristic polynomial is irreducible. For circulant matrices this is not the case.

It is easy to see that the *row-sum*, sum of all the elements in a row, is constant in a circulant matrix. This makes the row-sum an eigenvalue of the matrix. Since this eigenvalue belongs to the ground field, the only way to escape a discrete logarithm problem in the ground field is to make this eigenvalue, i.e., the row-sum 1. So the circulant matrix  $A$  should be chosen with the following properties:

- (ii) The matrix  $A$  has row-sum 1.
- (iii) The polynomial  $\frac{\chi_A}{x-1}$  is irreducible.

In the above case the security of the discrete logarithm problem in  $A$  is similar to that of the discrete logarithm problem in the finite field  $\mathbb{F}_{q^{d-1}}$ .

### 3.2 The discrete logarithm problem in $\frac{\mathbb{F}_q[x]}{x^d-1}$

Notice that

$$\frac{\mathbb{F}_q[x]}{x^d-1} \cong \frac{\mathbb{F}_q[x]}{x-1} \times \frac{\mathbb{F}_q[x]}{\psi(x)},$$

where  $\psi(x) = \frac{x^d-1}{x-1}$  and  $\gcd(d, q) = 1$ . So the discrete logarithm problem in  $\frac{\mathbb{F}_q[x]}{x^d-1}$  reduces into two different discrete logarithm problems, one in the field  $\mathbb{F}_q$  and the other in the ring  $\frac{\mathbb{F}_q[x]}{\psi(x)}$ . The matrix  $A$  can be chosen in such a way that the representer polynomial  $\phi_A(x) \bmod (x-1)$  is either 0 or 1 and hence reveals no information about the private key  $m$ . Note that:  $\phi_A(x) \bmod (x-1)$  is 0 or 1 is equivalent to the row-sum being 0 or 1 respectively. If the row-sum is zero the circulant matrix is singular; on the other hand if we assume the row-sum to be 1 (condition (ii)) then automatically  $\phi_A(x) \bmod (x-1) = 1$ .

If  $\psi(x)$  is irreducible, then the discrete logarithm problem is a discrete logarithm problem in the field  $\frac{\mathbb{F}_q[x]}{\psi(x)}$ . Hence the security of the discrete logarithm problem is the same as that of the discrete logarithm problem in  $\mathbb{F}_{q^{d-1}}$ .

The question remains, when is  $\psi(x)$  irreducible? We know that [2, Theorem 2.45],  $x^d - 1 = \prod_{d_1|d} \Phi_{d_1}(x)$ , where  $\Phi_k(x)$  is the  $k^{\text{th}}$  cyclotomic polynomial. It follows that if  $d$  is prime, then  $\psi(x) = \Phi_d(x)$ . Then the question reduces to, when is the  $d^{\text{th}}$  cyclotomic polynomial irreducible, for a prime  $d$ ? It is known [2, Theorem 2.47] that the  $d^{\text{th}}$  cyclotomic polynomial  $\Phi_d(x)$  is irreducible over  $\mathbb{F}_q$  if and only if  $q$  is primitive mod  $d$ .

We summarize the requirements on  $A$ , such that the discrete logarithm problem is as secure as the discrete logarithm problem in  $\mathbb{F}_{q^{d-1}}$ .

- (iv) The integer  $d$  is prime.
- (v)  $q$  is primitive mod  $d$ .

### 4 Why use the discrete logarithm problem with $d \times d$ circulant matrices over $\mathbb{F}_q$ instead of $\mathbb{F}_{q^d}$ ?

A quick answer to the above question is that multiplication in  $\mathcal{R}$ , which is isomorphic as an algebra to  $d \times d$  circulant matrices over  $\mathbb{F}_q$ , can be much faster!

In implementing the exponentiation in any group, the best known method is the famous *square-and-multiply* algorithm. Using *normal basis* [2, Definition 2.32], in a finite field of characteristic 2, squaring is cheap; it is just a cyclic shift of the bits. In our case, using Theorem 2.2, it is similar. How about multiplication?

The details of the complexity of multiplication is bit involved, but well studied. So we can skip the details here, and refer the reader to [6, 8]. The complexity of

multiplication in a finite field depends on the choice of basis. If one uses a normal basis, the least complexity is attained by an *optimal normal basis* [6, Chapter 5]. In that case, the complexity of multiplication in the field  $\mathbb{F}_{q^d}$  is  $2d - 1$  [6, Theorem 5.1], where  $\mathbb{F}_q$  is a field of characteristic 2. In the case of  $\mathcal{R}$ , that complexity reduces to  $d$  [8, Example 3]. In  $\mathcal{R}$  we get security of  $\mathbb{F}_{q^{d-1}}$ . So there is an obvious advantage of working with the circulant matrices than with a finite field – the complexity of computing the exponentiation reduces to almost half with only one extra bit.

Lastly, one can use the extended Euclidean algorithm to compute the inverse of a representer polynomial in  $\mathcal{R}$ . In an ElGamal like cryptosystem, one needs to compute that inverse. This will make decryption fast.

## 5 Conclusions

In this paper we study the discrete logarithm problem in the ring of circulant matrices. If the matrices are of size  $d$ , then we saw that under suitable conditions, the discrete logarithm problem is as secure as the discrete logarithm problem in  $\mathbb{F}_{q^{d-1}}$ . Since multiplying circulant matrices is easier, the discrete logarithm problem in circulant matrices is better than the discrete logarithm problem in a finite field  $\mathbb{F}_{q^{d-1}}$ .

There is not much history of looking at matrices for better (more secure) discrete logarithm problem. In this note the isomorphism of the circulant matrices with the algebra  $\mathcal{R}$  has reduced the central issue of this work to that of an implementation of finite fields. A way to look at  $\mathcal{R}$  and this study of the discrete logarithm problem in  $\mathcal{R}$  is, the finite field  $\mathbb{F}_{q^{d-1}}$  is embedded in  $\mathcal{R}$ . Though this is a valid way of looking at the present situation, it is not the whole view. For example, the issue with row-sum will not be transparent, unless one chooses to look at matrices. Also this opens up the possibility that there can be other matrices, *patterned matrices* or a subgroup of the circulant matrices, other than the one with row-sum 1; in which we can do much better with the discrete logarithm problem.

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