SPECTRAL THEORY FOR NON-NORMAL HILBERT SPACE OPERATORS

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Abstract. In this talk, we discuss the spectral theory for some non-normal operators on separable Hilbert spaces. Recall that a *Hilbert space* $\mathcal{H}$ is an inner-product space endowed with the inner-product $\langle \cdot, \star \rangle$, which is complete in the induced norm $\| \cdot \| \equiv \sqrt{\langle \cdot, \cdot \rangle}$. By the *Hilbert space operator* T on $\mathcal{H}$, we mean a linear transformation $T : \mathcal{H} \to \mathcal{H}$ satisfying

$$\|Th\| \leq M\|h\| \ (h \in \mathcal{H})$$

for some finite number $M > 0$. A linear operator $T^* : \mathcal{H} \to \mathcal{H}$ is said to be the *adjoint* of T if it satisfies

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \ (x, y \in \mathcal{H}).$$

A linear operator N on $\mathcal{H}$ is said to be *normal* if $N^*N = NN^*$. Multiplication operators on Hilbert spaces of square-integrable functions provide basic examples of normal operators. Any operator T on $\mathcal{H}$ for which $T^*T - TT^* \neq 0$ is *non-normal*. Any *isometry* (that is, an operator T satisfying $T^*T = I$) which is not surjective is a rather special example of a non-normal operator.

In the first half of this talk, we discuss one of the cornerstones of the Operator Theory: Spectral Theorem for Normal Operators. This theorem roughly asserts that any normal operator N can be obtained by integrating the co-ordinate function on the compact subset $\sigma(N)$ of the complex plane with respect to a "nice" projection-valued measure. Moreover, one can make sense out of $f(N)$ for a class of functions f which include, in particular, continuous functions. In the remaining half, we address the following delicate question:

What is the "size" of $T^*T - TT^*$ for a non-normal operator T?

There are two well-studied classes of non-normal operators for which the above question can be answered to a greater extent. In the case of hyponormal operators (operators for which $T^*T - TT^* \geq 0$), the most remarkable answer is due to Berger and Shaw ([2]). In the case of 2-isometries (operators for which $I - 2T^*T + {T^*}^2T^2 = 0$), the same is a recent achievement due to the speaker ([5]). Needless to say, these results have far-reaching consequences to the spectral theory for non-normal operators. The proof of the Berger-Shaw-type result for 2-isometries relies heavily on the basic theory of the so-called Cauchy dual operators, a subject of independent interest ([8]). We also plan to discuss some recent developments related to the Cauchy dual operators in the unbounded and multi-variable operator theory ([6] and [7]).

References

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