Assignment 8 - Metric Spaces, completions

MTH204, Spring 2017. IISER Pune.

In the following questions, let (X, d) denote a metric space. A sequence $x_1, x_2, ...$ in X will be denoted by the notation (x_n). Let $\mathcal{C}[X]$ denote the set of all Cauchy sequences in X.

1. Show that $(x_n) \sim (y_n)$ if $\lim_{n\to\infty} d(x_n, y_n) = 0$, defines an equivalence relation on $\mathcal{C}[X]$.
2. For $(x_n), (y_n) \in \mathcal{C}[X]$ show that $\big(d(x_n, y_n)\big)$ is a Cauchy sequence in $\mathbb{R}$. Define $d\big((x_n), (y_n)\big)$ (the distance between the Cauchy sequences) to be the limit of this Cauchy sequence.
3. For $(x_n) \in \mathcal{C}[X]$ denote its equivalence class in $\mathcal{C}[X]$ by $[(x_n)]$. Show that $d\big((x_n), (y_n)\big)$ depends only on the equivalence classes of (x_n) and (y_n) in $\mathcal{C}[X]$. Thus

   $$d\big([(x_n)], [(y_n)]\big) := d\big((x_n), (y_n)\big)$$

   is well defined.