Assignment 11 - Normed linear spaces

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1. Let V be a vector space and $|| \; || : V \to \mathbb{R}$ be a norm on it. Show that $d : V \times V \to \mathbb{R}$ defined by
$$d(v, w) = ||v - w||$$
gives a metric on V.

2. Let V be a normed linear space. Show that the norm $|| \; || : V \to \mathbb{R}$ is a continuous function.

3. For $p \geq 1$, define $|| \; ||_p : \mathbb{R}^n \to \mathbb{R}$ by
$$||(x_1, ..., x_n)||_p := \left(|x_1|^p + \cdots |x_n|^p\right)^{1/p}.$$
Except triangle inequality, verify that $|| \; ||_p$ satisfies the axioms of a norm on a vector space.

4. Let $V := \mathcal{C}[0, 1]$ denote the vector space of all real valued continuous functions on the interval $[0, 1]$. For $p \geq 1$, define $|| \; ||_p : V \to \mathbb{R}$ by
$$||f||_p := \left(\int_0^1 |f(x)|^p dx\right)^{1/p}.$$
Except triangle inequality, verify that $|| \; ||_p$ satisfies the axioms of a norm on a vector space.

5. Let $(V, || \; ||)$ be any normed linear space and let $\tilde{V}$ denote the completion of V with respect to the metric defined by $|| \; ||$. Show that $\tilde{V}$ has the structure of a complete normed linear space by :
    1. Defining addition and scalar multiplication.
    2. verifying axioms of vector space, e.g. distributivity of scalar multiplication over addition, associativity of addition etc.
    3. Define a norm on $\tilde{V}$. Briefly verify that it satisfies the properties of a norm.