Recall

Recall from the previous lecture

- Rational numbers have recurring decimal expansion
- Real numbers are numbers which have infinite decimal expansion which may or may not be recurring.
- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.
- An **infinite sequence** $x_1, x_2, x_3, ...$ where x_n is a real number for every natural number n.
- **Absolute value** $|x|$ of a real number x.
- Distance between two real numbers x and y , this is just $|x - y|$.

Lecture 2- Convergent Sequences and limits

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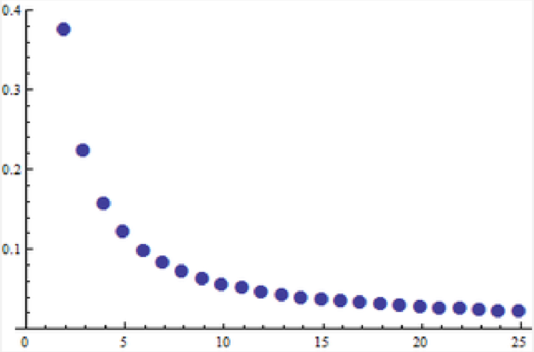
Outline of this lecture

Consider an infinite sequence $x_1, x_2, x_3, ...$

How many of us understand the phrase
$|x_n|$ *becomes smaller and smaller as n becomes large*?



Please pay attention to the following concepts

- Convergent sequence.
- Limit of a convergent sequence.

The concept of convergence and limits lies at the heart of calculus.

The sequence $1/n$

We are now going to build a somewhat complicated definition by studying a simple example.
Consider the infinite sequence

$$x_1, x_2, x_3, \ldots.$$

where

$$x_n = \frac{1}{n}.$$

The terms of this sequence approach 0, i.e. they become smaller and smaller as n grows larger.

Goal

Give a precise meaning to the phrase *become smaller and smaller*.

As small as you want

Let us arrive at a precise definition by asking some questions.

Q: What does it exactly mean when one says x_n approaches zero?

A: It means you can make $|x_n|$ as small as you want.

Q: Can you make it smaller than 0.4?

A: Yes ! note that $1/3 = 0.33 < 0.4$. Moreover $n \geq 3$

$$|x_n| = \frac{1}{n} < 0.4.$$

Going to larger stage to make it smaller

Q: How about making it smaller than 0.001

A: For $n \geq 1001$
$$|x_n| \leq 0.001.$$

Q: Can you make it smaller than 10,000?
A: This is easy. $|x_n| \leq 1 < 10,000$ for all $n \geq 1$.

Observe the following:

- To make $|x_n|$ smaller than than 0.4, we had to restrict to $n \geq 3$.
- But to make it smaller than 0.001 we had to look at $n \geq 1001$.
- Depending on how small you want to make the terms, we may have to start at a larger n.

Watch the precise definition evolve

Intuitive definition

A sequence x_n approaches zero if $|x_n|$ can be made as small as you like after a certain stage

Precise Definition

A sequence x_n converges to zero if
for any $\epsilon > 0$
there exists N, such that
$|x_n| < \epsilon$ for all $n \geq N$.

Example 1

Let us apply this definition to the example the sequence $x_n = 1/n$.

Q: Take $\epsilon = 0.4$. Can you make $|x_n| < 0.4$?
A: Yes ! Let $N = 3$. for all $n \geq N$, $|x_n| < 0.4$.

Q: Take $\epsilon = 1$ million. Can you make $|x_n| < 1$ million?
A: Easy. All terms are already less than 1. So $N = 1$ will work. For every $n \geq 1$

$$|x_n| < 1 \text{ million}$$

Q: Take $\epsilon = 0.001$. Can you make $|x_n| < 0.001$ after a certain stage?
A: $N = 1001$ works.

Example 2:

Let us slightly modify the earlier sequence by making the first 100 terms large. Consider the new sequence

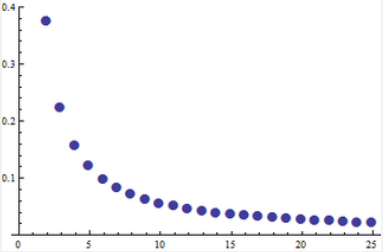
$$x_n = \begin{cases} 10,000 & \text{if } \; n \leq 100 \\ 1/n & \text{if } \; n > 100 \end{cases}$$

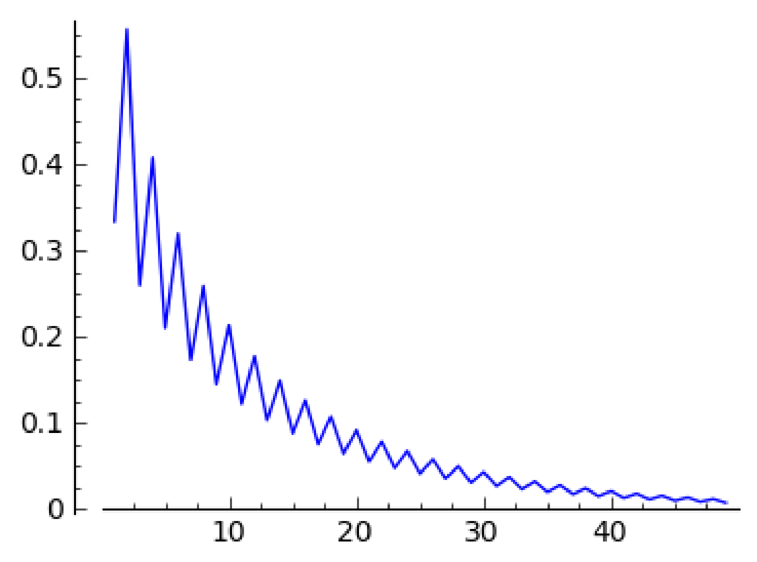
Q: Take $\epsilon = 0.4$. Can you make $|x_n| < 0.4$ after a certain stage ?
A: Now, $N = 3$ will not work. Because $|x_n| = 10,000$ for all n up to 100. So we have to take N at least 101. $N = 101$ works.

Q: $\epsilon = 1$ million?
A: Think !

Q: $\epsilon = 0.001$. Can you make $|x_n| < 0.001$ after a certain stage?
A: Again $N = 1001$ will work.

- Changing the first few terms of a sequence does not affect convergence.
- i.e. if x_n converges to 0, then after changing finitely many terms the new sequence will also converge to 0.
- We may have to choose a larger value of N for some ϵ. However the modified sequence will still converge to zero.




0.5
0.4
0.3
0.2
0.1
0
10
20
30
40

Example 3 (constant sequence)

Consider the constant sequence

$$0,\ 0,\ 0,\ 0, \ldots.$$

i.e. $x_n = 0$ for all natural numbers n.

Q: Take $\epsilon = 0.00001$. Can you make $|x_n| < 0.0001$ after a certain stage?
A: $|x_n|$ is always equal to 0 in this case. Thus for any value of $\epsilon > 0$,

$$|x_n| < \epsilon \text{ for all } N$$

So $N = 1$ works. Note that any other natural number also works.

Example 4

$$x_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

Q: Take $\epsilon = 20$. Can you make $|x_n| < 20$ after a certain N?

A: For all terms x_n, $|x_n|$ is either 0 or 1. So it is already less than 20. Any N will work, e.g. $N = 1$.

Q: Take $\epsilon = 0.5$. Can you make $|x_n| < 0.5$?
A: No. There is no N such that after the N-th stage $|x_n| < 0.5$. This is because for any such N, there is always an even number n bigger than N. e.g. $n = 2N$.

$$|x_{2N}| = 1 \not< 0.5.$$

This sequence does not converge to 0.

To show this we produced an ϵ (=0.5) for which no N works.

Consider the sequence

$$x_n = 0 \text{ for all } n.$$

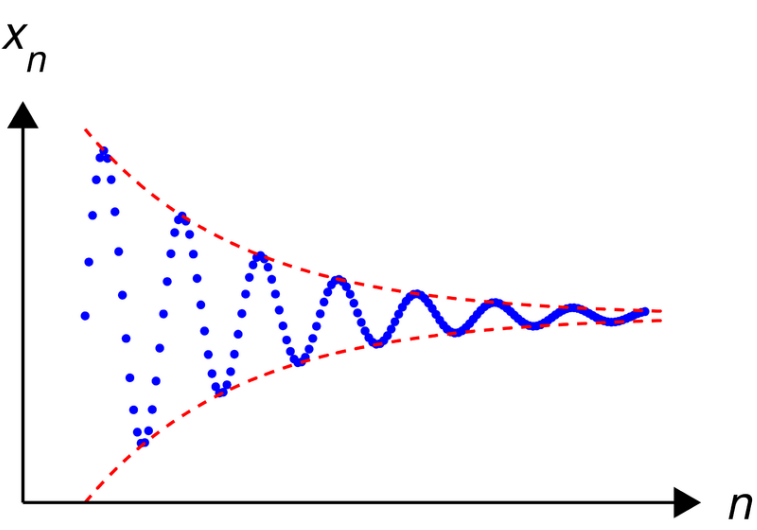
Convergent sequence and its limit

An infinite sequence $x_1, x_2, ..., x_n$ is said to converge to a real number x if the distance between x_n and x becomes smaller and smaller as n becomes larger.

Convergent sequence

A sequence x_n **converges to** x if,
given any $\epsilon > 0$,
there exists N such that
$|x_n - x| < \epsilon$ if $n \geq N$.

- x is called the **limit** of the sequence x_n.


x_n
n

Example 5:

$$x_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 1 + 1/n & \text{if } n \text{ is even} \end{cases}$$

The idea is $1/n$ approaches zero as n becomes large, so $1 + 1/n$ will approach 1.

Q: Can $|1 - x_n|$ made smaller than 0.01 for all n after a particular stage?
A: In the language of the definition, we are given $\epsilon = 0.01$ and we want to find a natural number N such that all terms after N are at a distance less than the given ϵ from 1.

Choose $N = 200$.
IF $n > N$, $1/n < 1/200 < 0.01$.
$|1 - x_n| = |1/n| < 0.01$

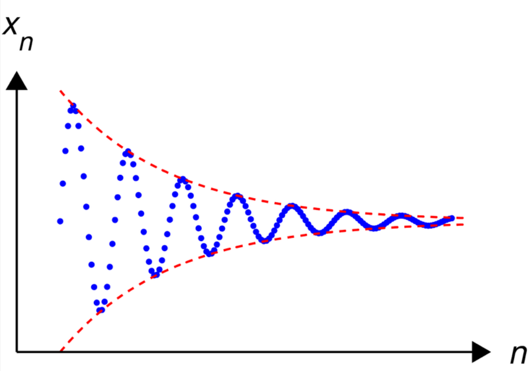
We did not choose N very carefully. In fact $N = 101$ would also work.
It is not important here to choose an optimal N.

Example 6

The sequence

$$x_n = n$$

does not converge to any limit.
Since its terms grow larger and larger, the sequence is not of the type show in the figure below:



Example 7

$$x_n = \frac{2n}{n^2+1}$$

n^2+1 grows more rapidly than $2n$. Therefore the above sequence converges to zero. To prove this we need to show that we can make the terms smaller than a given ϵ, we need to ensure:

$$\frac{2n}{n^2+1} < \epsilon$$

$$i.e. \quad 2n < \epsilon(n^2+1)$$

$$i.e. \quad n + 1/n > \frac{2}{\epsilon}$$

Since $1/n$ is always positive, the above will hold if

$$n > \frac{2}{\epsilon}$$

Q: What are the possibilities for N in this case?

Assignment I

1. Define what do you mean by '1 is a limit point of an infinite sequence x_n'.
2. Show that $\frac{1}{n^2}$ converges to zero.
3. Show that the sequence $x_n = 2 - 1/n^2$ converges to 2.
4. Consider the sequence $x_n = \frac{n}{2n+1}$. Does x_n converge to any limit? Explain.
5. Consider the sequence $x_n = \frac{n^2}{n+1}$. Does x_n converge to any limit? Explain.
6. Give an example of a sequence which does not converge anywhere (other than the above sequences and examples discussed in the class).