Single variable calculus

The course is designed assuming you know only school level mathematics.

We will **never** assume exposure to 11th/12th standard mathematics.

Weekly structure

Every week,

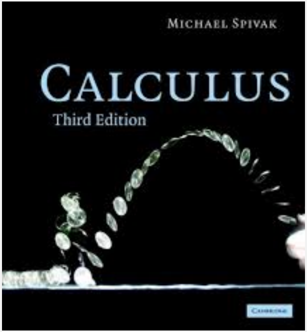
- **Lectures**: there will be 2 lectures and 1 tutorial.
- **Assignments**: there will be about 5 to 10 assignment/homework problems every week. These problems will usually be based on examples done in the class. You are not required to submit the homework and it will not be graded.
- **Quizzes/Tutorials**: In the first 15 minutes of every tutorial (from 2nd week) there will be a Quiz based on examples done in class and the homework problems.
- **Self Study**: you are expected to spend **at least 4 hours** on self study.

Grading scheme

- Midsem and Endsem carry 35% weightage each.
- Quizzes carry 30% weightage.
- You get a zero in a quiz if you are absent.
- There will be no repeat quizzes.
- Worst 2 quizzes will be ignored.

Reference Book

Main reference book : **Calculus** by *M. Spivak*. (3rd edition).



Course webpage (for notes and assignments)

- A webpage for this course will be maintained at

www.iiserpune.ac.in/∼amit/101

- All slides shown in the class will be uploaded on the course webpage.

Office hours

After the first tutorial, information on office hours will be displayed in the class as well as on the course web page.

Objectives of this course

- To explain the basic concepts of calculus , for example
  - limits
  - derivatives
  - integrals.
- To discuss applications of single variable calculus.
- To serve as an introduction to undergraduate level mathematics.

Lecture 1- Real Numbers

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Outline of this lecture

Today, we will **try** to add two numbers.

Please pay attention to the following concepts in this lecture

- Real numbers
- An infinite sequence of real numbers

The set of real numbers is the most important set in this course.

The sets $\mathbb{N}$ and $\mathbb{Z}$

- The set $\{1, 2, 3, 4, ...\}$ is called the set of natural numbers.
- This set will be denoted by $\mathbb{N}$.
- We enlarge this set by including 0 (zero) and the negative integers. The resulting set
$$\{0, 1, -1, 2, -2, 3, -3, ...\}$$
is called the set of integers. It will be denoted by $\mathbb{Z}$.
- We can also think of $\mathbb{Z}$ as a set extending infinitely in both directions and write it as
$$\{..., -3, -2, -1, 0, 1, 2, 3, ...\}.$$

We all know how to add and multiply integers.

The set $\mathbb{Q}$

- We can further enlarge the set of integers $\mathbb{Z}$ by including fractions

$$\frac{1}{2}, \frac{3}{4}, \frac{5}{7}, \frac{101}{2}, \textit{etc.}$$

- Consider the set of all fractions $\frac{m}{n}$ where m and n are integers and $n \neq 0$.
- We declare $\frac{m_1}{n_1} = \frac{m_2}{n_2}$ if $m_1 n_2 = m_2 n_1$.
- The set of all such fractions is called the set of rational numbers. It is denoted by $\mathbb{Q}$.

Question

How does $\mathbb{Z}$ sit inside $\mathbb{Q}$?

Answer

Every integer n can be thought of as the fraction $\dfrac{n}{1}$. Thus

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}.$$

Addition and multiplication of rational numbers

We can add and multiply rational numbers using the formulae

$$\frac{m_1}{n_1} + \frac{m_2}{n_2} = \frac{m_1 n_2 + m_2 n_1}{n_1 n_2}.$$

$$\frac{m_1}{n_2} \times \frac{m_2}{n_2} = \frac{m_1 m_2}{n_1 n_2}.$$

Decimal expansion of rational numbers

Every rational number has a decimal expansion. For example:

- $1/2 = 0.5$
- $-5/4 = -1.25$
- $1/30 = 0.033333333\cdots \quad = 0.0\overline{3}$
- $-5/7 = -0.714285714285\cdots \quad = 0.\overline{7142857}$
- $3227/5550 = 0.58144144144144\cdots \quad = 0.58\overline{144}$

We put a bar on the repeating decimals.

The decimal expansions in the last three examples is called **recurring**. Since

$$0.5 = 0.500000000\cdots \to \text{ (0 repeats `forever')}$$

The decimal expansion in the first two examples are also recurring.

Recurring decimal expansion

Theorem

The decimal expansion of every rational number is recurring.

Question

What about ‘numbers’ whose decimal expansion is not recurring?

For many purposes, especially for modeling real world phenomenon, rational numbers are not sufficient. We need real numbers.

A way to think of real numbers

Definition

Real numbers are ‘numbers’ with decimal expansions which may or may not be recurring. The set of real numbers is denoted by $\mathbb{R}$.

- Every rational number is a real number.
- There are plenty of real numbers which are not rational.

Definition

A real number which is not rational is called **irrational**. Irrational numbers are precisely those real numbers which do not have recurring decimal expansion.

Example

There is a real number $\sqrt{2}$ such that $\sqrt{2} \times \sqrt{2} = 2$.

$$\sqrt{2} = 1.41421356237309504880168872420969807856967187 5 \cdots \rightarrow$$

$\sqrt{2}$ is irrational ! (you will see this in MTH100).

It is impossible to 'know' the complete decimal expansion of $\sqrt{2}$ or any other irrational number!

$$1.\overline{9} = 1.99999999\cdots = 2.$$

Rule

whenever there is a recurring 9 we will change the previous decimal by +1

For example

- $1.2349999999999\cdots = 1.2345$
- $0.2999999999\cdots = 0.3$
- $5499.99999999\cdots = 5500$
- and so on...

If one follows this rule, one gets a unique decimal expansion for a real number.

Question

How to add two real numbers using their decimal expansion?

Particular example

What is

$$0.\overline{5} + 0.\overline{6} = ?$$

The usual technique of addition is to start from the right-most decimal and carry-over whenever the result of addition is more than 10.

However in this case where the decimal expansion is infinite, there is no right-most decimal !

In this particular example we already know

$$0.\overline{5} = 5/9$$

$$0.\overline{6} = 2/3$$

$$\frac{5}{9} + \frac{2}{3} = \frac{15 + 18}{27} = \frac{33}{27} = \frac{11}{9} = 1.\overline{2}$$

Answer

$$0.\overline{5} + 0.\overline{6} = 1.\overline{2}$$

Can we find this answer, just looking at decimal expansions?

This will allow us to add real numbers which are not rational.

An idea...

For the real number $0.\overline{5}$ define the following

- $x_1 = 0.5$
- $x_2 = 0.55$
- $x_3 = 0.555$
- $x_4 = 0.5555$
- and so on....

We think of each x_n as a number which approximates $0.\overline{5}$.

This approximation gets better and better as n becomes large.

Infinite sequence

Definition

An infinite sequence of real numbers is a sequence

$$x_1, x_2, x_3, ...,$$

where each x_n is a real number.

Thus in the previous slide, we have constructed a sequence

$$x_1, x_2, x_3, ...$$

which approaches $0.\overline{5}$.
Similarly we construct a sequence which approaches $0.\overline{6}$.

- $y_1 = 0.6$
- $y_2 = 0.66$
- $y_3 = 0.666$
- and so on...

We already know how to add $x_n + y_n$

- $z_1 = x_1 + y_1 = 0.5 + 0.6 = 1.1$
- $z_2 = x_2 + y_2 = 0.55 + 0.66 = 1.21$
- $z_3 = x_3 + y_3 = 0.555 + 0.666 = 1.221$
- $z_4 = x_4 + y_4 = 0.5555 + 0.6666 = 1.2221$
- $z_5 = x_5 + y_5 = 0.55555 + 0.66666 = 1.22221$
- $z_6 = x_6 + y_6 = 0.555555 + 0.666666 = 1.222221.$

By inspection we see that as n gets larger
the sequence 'z_n approaches $1.22222222\cdots = 1.\overline{2}$ as n gets larger and larger'. Thus $1.\overline{2}$ is the answer.

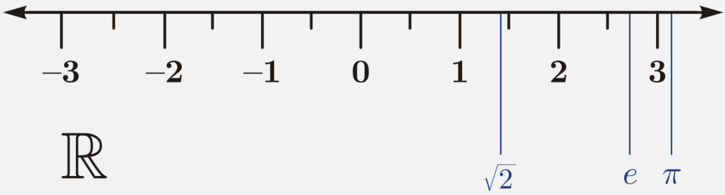
> What is the precise meaninf of '**a sequence approaches a real number**'?

We will see this in the next lecture.

- Similar idea can be used to add any two real numbers.
- Let us now assume without proof that one can add, multiply and divide real numbers, just like we do for rational numbers.
- Remember we never divide by zero !

How to visualize $\mathbb{R}$?

One must always visualize $\mathbb{R}$ as a line extending infinitely in both directions.



The above line will be referred to as the **Real Line**.

Absolute value

Definition

Absolute value For a real number x, we define $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. $|x|$ is called the **absolute value** of x. It should be thought of as the distance of x from 0.

Similarly, $|x - y|$ is the distance between x and y.

For example:

- $|-1-1| = |-2| = 2$. Thus distance between -1 and $+1$ is 2.
- $|-4| = |4| = 4$

Summary

You should remember/understand the following points from this lecture in order to understand the next

- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- Every rational number has a recurring decimal expansion.
- $\mathbb{R}$ = the set of real numbers (numbers whose decimal expansion may or may not be recurring).
- An infinite sequence of real numbers. In particular, remember the sequence
$$0.5,\ 0.55,\ 0.555,\ 0.5555,\ \ldots$$
approximating $0.\overline{5}$.
- Absolute value of real numbers.
- The Real Line.