12 Feb 2018 ①

ENHANCING STIMULATED EMISSION

Stimulated emission, which is a necessary consequence of interaction of radiation with atoms can be exploited to construct an oscillator ~~of~~ or amplifier of radiation at the resonant freq ω_{21}.

Consider a collection of atoms exposed to (broad-band) radiation. Let these be 2-level atoms for simplicity. Then the rate at which the energy density, or intensity of the radiation changes is governed by the equations

$$\left.\frac{d\rho}{dt}\right|_{abs} = -N_1 \hbar\omega_{21} R_{12}$$

$$\left.\frac{d\rho}{dt}\right|_{st.em} = +N_2 \hbar\omega_{21} R_{21}$$

NOT IMP

$$\left.\frac{d\rho}{dt}\right|_{sp.em} = +N_2 \hbar\omega_{21} R'_{21}$$

– N_2: upper level pop. N_1: lower level pop.

– Of these three, only R_{21} & R_{12} depend on ρ

RECAP: $I(\omega) = c\rho(\omega) = c\hbar\omega(n/V)$

$$= \frac{1}{2}\epsilon_0 \mathcal{E}^2(\omega)$$

$$\sigma_{st.}(\omega) = \hbar\omega_{21} R_{21} / \hbar\omega$$

The average rate of change of density of the radiation per unit volume of the medium exposed to radiation is

$$\frac{d\rho}{dt} = \sigma I (N_2 - N_1)$$

or

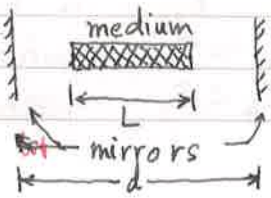
$$\frac{dI}{dz} = \frac{d\rho}{dt} = \sigma I (N_2 - N_1) \Rightarrow I = I_0 e^{-\alpha z}$$

where $\alpha = (N_1 - N_2)\sigma$

$[\alpha(\omega)]$ $[\sigma(\omega)]$

If $N_2 > N_1$, the density of radiation is enhanced, and if $N_1 > N_2$ the density is depleted. — we have therefore amplification or attenuation. If the medium has length L, then over one return trip of ~~the~~ a radiation pulse, the gain (or loss) will be given as the ratio

$$G = \frac{I(\omega, 2L)}{I(\omega, 0)} = e^{-2\alpha(\omega)L}$$



RECALL Einstein A, B coefficients

Spontaneous emission depopulates N_2 so that thermal equilibrium is maintained. Thus, due to spontaneous emission, we always have

$$N_2 / N_1 = \exp(-\hbar\omega_{21}/KT) \quad \text{and} \quad \alpha = N_1 - N_2 \text{ is +ve}$$

irrespective of the value of R_{21} [or σ_{21}]

Thus $G = \exp[-2\alpha L]$ is always < 1 for a 2-level system. Amplification cannot be achieved by a 2-level system.

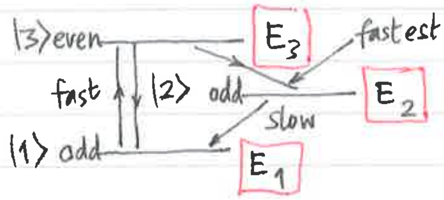
A THREE LEVEL SYSTEM

Suppose we have a three-level system which is such that

$A_3 > A_2 > A_1$, i.e. $\tau_3 < \tau_2 < \tau_1$ preferably $\tau_3 \ll \tau_2$

and R_{23}, R_{13} are non-zero

and R_{12} is small



then a "population inversion" may be obtained between levels 2, 1 if $>50\%$ of N_1 is transferred to N_3 by stimulated absorption and $|2\rangle$ is rapidly populated by decay of $|3\rangle$ rather than $|3\rangle$ decaying to $|1\rangle$

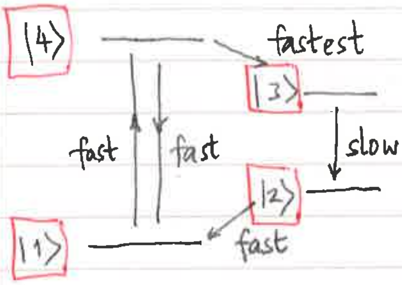
⇒ R_{32} is very high (dipole allowed) while R_{21} is weak (dipole forbidden)

⇒ R_{13} or R_{31} is high (dipole allowed) to ensure population pumping

⇒ Difficult to meet all conditions simultaneously.

A FOUR LEVEL SYSTEM

A system such that $E_1 \rightarrow E_4$ is a fast dipole allowed transition and $E_3 \rightarrow E_2$ is a slow dipole forbidden transition is a more practical system for achieving population inversion.



⇒ R_{43} is high

R_{21}, ~~R_{12}~~ is high

R_{41}, R_{14} are high

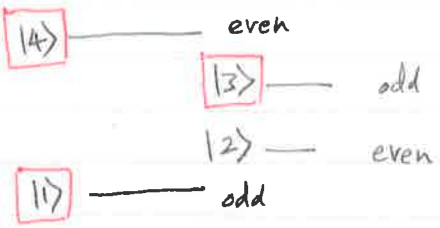
R_{32} is low.

$\tau_3 > \tau_2$, preferably $\tau_3 \gg \tau_2$

$\tau_4 < \tau_3$, preferably $\tau_4 \ll \tau_3$

12 Feb 2018 3

For a four level system to achieve inversion the parities of the 4 levels must be of the type if all transition are radiative.



Ensures R_{41} R_{43} R_{21} requirements
But this scheme conflicts with the requirement that R_{32} should be slow.
⇒ One of the fast transitions R_{43} ~~or~~ has to be non-radiative.

THRESHOLD CONDITION FOR LASING

We defined $\alpha(\omega) = (N_1 - N_2)\,\sigma(\omega)$ in which $|1\rangle$ and $|2\rangle$ are levels between which lasing occurs.

LASING : enhanced (amplified) emission of radiation at a particular ω and $\hat{\varepsilon}$ due to stimulated absorption at the same ω and $\hat{\varepsilon}$

SPONTANEOUS & STIMULATED emission rates are equal in a radiation field that contains on an average 1 photon per mode

Ratio of stimulated to spontaneous emission rates gives the number of photons in that mode → we wish to enhance / amplify this.

However, the more the upper level population, the greater is also the absolute value of spontaneous emission rate, thereby reducing the population inversion.

At thermal equilibrium $N_2 < N_1$, but if inversion has been achived by some means $N_1 < N_2$, or strictly $N_1 \frac{g_2}{g_1} < N_2$

If the last condition is satisfied $\alpha < 0$ (inversion condition)
and $G(\omega) = \exp(-2\alpha L)$ is positive

There may be
Attenuation / diffraction / reflection losses : $I(2d)/I(0) = e^{-\gamma}$
where d is the mirror separation

Then $G(\omega) = \dfrac{I(\omega, 2d)}{I(\omega, 0)} = e^{-2\alpha L - \gamma}$

If $G(\omega) = 1$ we have a stable oscillator, all losses are overcome

Although $N_2 > N_1$, N_2/N_1 is not a constant

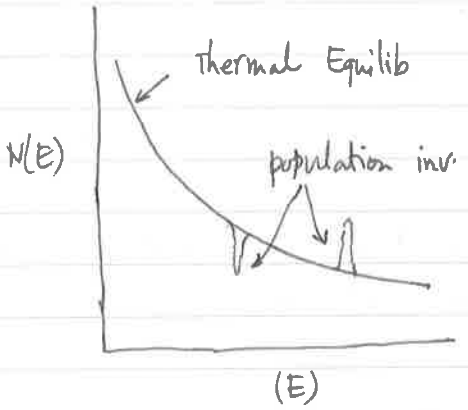
REASON: Increase in photon numbers increases stimulated emission
stimulated emission reduces population inversion

RESULT: N_2/N_1 settles to an equilibrium condition in a short time

For $G(\omega) = 1$, $2\alpha L = -\gamma \Rightarrow 2(N_1 - N_2)\sigma_L = -\gamma$

$$\Rightarrow \Delta N_{thresh} = \frac{\gamma}{2\sigma L}$$

$\Delta N > \Delta N_{thresh}$ makes the oscillator an amplifier



12 Feb 2018 (5)

RECAP: The rates for spontaneous and stimulated emission are

stim: $$\left\{\begin{matrix} R_{21} \\ R_{12} \end{matrix}\right. = \frac{\pi e^2}{3\epsilon_0 \hbar^2} \rho(\omega_{21}) \langle \vec{r}_{21} \rangle^2$$

[unpolarised $\hat{\mathcal{E}}$]
[for polarised replace $\vec{r}$ by $3\vec{z}$]
or $3\hat{\mathcal{E}}\cdot\vec{r}$

spont: $$R'_{21} = \cancel{\frac{4}{3}\frac{e^2}{8\pi^2 \hbar c^3 \epsilon_0} \omega_{21}^3 \langle r_{12} \rangle^2}$$

$$R'_{21} = \frac{e^2}{3\pi\hbar\epsilon_0 c^3} \omega_{21}^3 \langle \vec{r}_{12} \rangle^2$$

The key quantity here is the matrix element $\langle \vec{r}_{12} \rangle$. This element has radial and angular integrals:

$$\langle \vec{r}_{12} \rangle = \int \cancel{R_1(r_1) A_1(\theta_1, \phi_1)}$$

$$= \int [R_1(r)\; A_1(\theta,\phi)]^* \vec{r}\; [R_2(r)\; A_2(\theta,\phi)]\; d^3\vec{r}$$

The radial part of the integral is generally non-zero, but the angular part is non-zero only for certan combinations of the angular momentum quantum numbers.
This condition gives rise to selection rules.

$\hat{\mathcal{E}}$ can be written in spherical components

$$\mathcal{E}_{\pm 1} = \frac{1}{\sqrt{2}}(\mathcal{E}_x \pm i\,\mathcal{E}_y) \qquad \mathcal{E}_0 = \hat{\mathcal{E}}_z$$

If $\vec{k} = k_z \hat{z}$, then $\mathcal{E}_{\pm 1}$ are circular polarisation components

Then $$\hat{\mathcal{E}}\cdot\vec{r} = \sum_q \mathcal{E}_q^* I^q_{\{1\}\{2\}}$$

$$I^q_{\{1\}\{2\}} = \int R_{n_1 \ell_1}(r) R_{n_2 \ell_2}(r)\, Y_{\ell_1 m_1}(\theta,\phi)\, Y_{\ell_2 m_2}(\theta_2,\phi_2)\, r^3\, dr\, d\Omega$$

From this we get the following rules for dipole transitions

Parity : must change

magnetic quantum number : $\Delta m = 0$ for linear pol. $\hat{\varepsilon} \parallel \hat{z}$

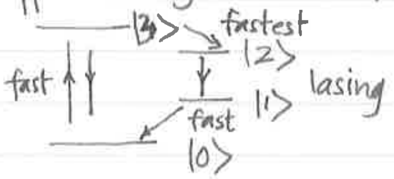
$\Delta m = \pm 1$ fo circular pol $\hat{k} \parallel \hat{z}$

orbital angular momentum : $\Delta \ell = \pm 1$

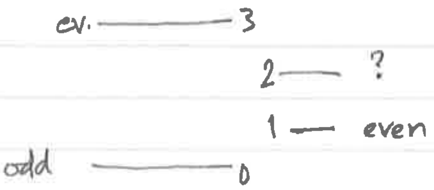
total angular momentum $\Delta j = 0, \pm 1$ but not $j=0 \rightarrow j=0$

Although a 3-level laser is in principle a workable laser, it puts a serious demand on pumping from $|1\rangle$ to $|3\rangle$ to achieve population inversion between $|2\rangle$ and $|1\rangle$, the lasing levels

An alternative is a four level laser, in which the ground state $|0\rangle$ and the first excited states $|1\rangle$ are considerably apart ($\gg kT$) and a fast pump is possible from $|0\rangle$ to $|3\rangle$, where $|3\rangle$ is above the upper lasing level $|2\rangle$



$|1\rangle$ & $|0\rangle$ have to be of opp parities,
$|3\rangle$ & $|0\rangle$ also "



$\tau_3 < \tau_2$ or even better $\tau_3 \ll \tau_2$
$\tau_2 > \tau_1$ or even better $\tau_2 \gg \tau_1$

Thus $|2\rangle$ & $|1\rangle$ have to be of even parity if $|2\rangle \to |1\rangle$ is to be weakly forbidden to ensure inversion

If $|2\rangle$ is even, $|3\rangle \to |2\rangle$ can be rapid only if it is non-radiative.

Thus, for a "real" 4-level system, the upper lasing level must be populated non radiatively.
A 4-level laser has the advantage that the lower lasing level is naturally empty, unlike the 3-level system.

RATE EQUATIONS FOR A 4 LEVEL SYSTEM.

Assume $|3\rangle$ is pumped from $|0\rangle$ with some rate R and that $|3\rangle$ fills up (pumps into) $|2\rangle$ at some pump rate 'P'

Assume $|2\rangle$ radiates to $|1\rangle$ ~~by~~ with Einstein coeff B_{21} and has other loss rate L_2 (similarly for $|1\rangle$)

Then the rate eqns read

$$\frac{dN_1}{dt} = -N_1 B_{12}(n\hbar\omega) + N_2 A_{21} - N_1 L_1 + N_2 B_{21}(n\hbar\omega)$$

$$\frac{dN_2}{dt} = P - N_2 B_{21}(n\hbar\omega) - N_2 A_2 + N_1 B_{12}(n\hbar\omega) - N_2 L_2$$

$\{A_{22} \equiv A\}$
$\{B_{21} = B_{12} \equiv B\}$
$\{\Delta N = N_2 - N_1\}$

$$\therefore \frac{dN_1}{dt} = \Delta N\, B(n\hbar\omega) - N_1 L_1 + N_2 A \qquad (1)$$

$$\frac{dN_2}{dt} = P - \Delta N\, B(n\hbar\omega) - N_2 L_2 - N_2 A \qquad (2)$$

$$\frac{dn}{dt} = -\gamma n + N_2 B_{21}(n\hbar\omega) - N_1 B_{12}(n\hbar\omega)$$

$$\frac{dn}{dt} = -\gamma n + \Delta N B(n\hbar\omega) \qquad (3)$$

Under steady conditions each rate is zero

$$\frac{dN_1}{dt} = \frac{dN_2}{dt} = \frac{dn}{dt}$$

Adding first two we get $\quad P = N_1 L_1 + N_2 L_2$

Adding second & third $\quad P = \gamma n + N_2 A + N_2 L_2$

Multiply (1) by L_2, (2) by L_1

$$(\Delta N\, B(n\hbar\omega) + N_2 A - N_1 L_1) L_2 = 0$$
$$(P - \Delta N\, B(n\hbar\omega) - N_2 A - N_2 L_2) L_1 = 0$$

$$\Delta N = \frac{(L_1 - A)P}{Bn\hbar\omega\,(L_2 + L_1) + AL_1 + L_1 L_2}$$

For inversion L_1 (lower state loss rate) $>$ A (upper state spontaneous ~~ste~~ em rate)