

ANS 1

de Broglie wavelength  $\lambda = h/p$

for an orbit containing integer number of wavelengths,

$$2\pi r = n\lambda = nh/p = nh/mv$$

$$\Rightarrow \underline{mvr = \frac{nh}{2\pi}} \quad \left. \vphantom{\frac{nh}{2\pi}} \right\} \begin{array}{l} \text{angular momentum} \\ \text{is quantised} \end{array} \quad [1]$$

From classical mechanics, we have for this circular motion

$$\frac{mv^2}{r} = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2} \quad [2]$$

(centripetal force = electrostatic force)

$$\text{and} \quad KE = \frac{1}{2}mv^2 \quad PE = \frac{-e^2}{4\pi\epsilon_0 r}$$

$$\begin{aligned} \text{Using [2], total energy} &= KE + PE \\ &= \frac{-e^2}{4\pi\epsilon_0 r} \end{aligned} \quad [3]$$

$$\text{Using [1]} \quad \underline{r = n \frac{h}{2\pi} \frac{1}{mv}}$$

$$\text{Using [2] this becomes} \quad r = \frac{n^2 h^2 \epsilon_0}{\pi m e^2}$$

Put this value of  $r$  in [3] to get

$$\underline{E = \frac{-me^4}{8\epsilon_0^2 h^2} \frac{1}{n^2}} \quad \left. \vphantom{\frac{1}{n^2}} \right\} \begin{array}{l} \text{required form,} \\ E \propto \frac{1}{n^2} \end{array}$$

ANS 2

$$\psi_{2p} = \frac{1}{4\sqrt{2\pi}} \left(\frac{1}{a_0}\right)^{3/2} \left(\frac{r}{a}\right) e^{-r/2a} \cos\theta$$

This corresponds to  $n=2, l=1, m=0$ ;  $m = \pm 1$  will be similar

$$\left\langle \frac{3\cos^2\theta - 1}{r^3} \right\rangle = \left\langle \psi_{2p} \left| \frac{3\cos^2\theta - 1}{r^3} \right| \psi_{2p} \right\rangle \quad \text{--- (I)}$$

$$|\psi(0)|^2 = |\psi_{2p}(r=0)|^2 = 0$$

The integral I is given by

$$\begin{aligned} &= \int \frac{1}{32\pi a^5} r^2 e^{-r/a} \cos^2\theta \frac{3\cos^2\theta - 1}{r^3} \cdot r^2 dr \sin\theta d\theta d\phi \\ &= \frac{1}{32\pi a^5} \int_0^\infty r e^{-r/a} dr \cdot \int_0^\pi [3\cos^4\theta \sin\theta - \cos^2\theta \sin\theta] d\theta \cdot \int_0^{2\pi} d\phi \\ &= \frac{1}{32\pi a^5} \cdot \underbrace{a^2} \cdot \underbrace{\frac{8}{15}} \cdot \underbrace{2\pi} \\ &= \frac{1}{30a^3} \end{aligned}$$

$$\Rightarrow \Delta v = A \left[ \frac{8\pi}{3} \cdot 0 + \frac{1}{30a^3} \right]$$

$$= \frac{A}{30a^3}$$

ANS 3

This can be worked out in terms of either the binding energies calculated to the level of the fine structure or by obtaining the average distances of the electrons in the 2s & 2p states relative to the 1s electron.

(It is not enough to say that the 2p electron has a preferred angular orientation and hence has less screening)

We can show that with any value of the nuclear charge, in the hydrogenic configuration:

$$\langle r \rangle_{1s} < \langle r \rangle_{2s} < \langle r \rangle_{2p}$$

- Thus the 2s electron lies on the average closer to the 1s electron and the nucleus, than the 2p electron.
- So a fair portion of the 2s electron distribution lies inside the average radius of the 1s electron distribution. Note the  $\psi_{2s}$  &  $\psi_{1s}$  are both large, non-zero near  $r=0$ .
- On the other hand the 2p electron mostly lies beyond  $\langle r_{1s} \rangle$  and has zero density  $\psi_{2p}$  is small near  $r=0$
- Hence the  $\psi_{1s}$  electron effectively smears out the nuclear effect for a large part of the 2p distribution, but not much for the 2s.

⇒ Screening is more effective for 2p; i.e.  $(Z-\sigma)$  is smaller for 2p than for 2s.

⇒ Check in terms of energy levels (in  $\text{cm}^{-1}$ )

1s2p	171134	1p
	169087	3p
1s2s	166277	1s
	159855	3s
1s <sup>2</sup>	0	1s

$\text{cm}^{-1}$

ANS 4

Screening of the second electron is quantified by screened nuclear charge  $Z - \epsilon = Z_{\text{eff}} \equiv 2 - \epsilon$

This is obtained from the binding energy consideration

$$E = -2.90 \text{ a.u.} = -\frac{1}{2} Z^2 - \frac{1}{2} Z_{\text{eff}}^2$$

$$\Rightarrow -2.90 \text{ a.u.} = -\frac{1}{2} Z^2 - \frac{1}{2} Z_{\text{eff}}^2$$

$$\Rightarrow \underline{Z_{\text{eff}}^2 = 1.8}$$

Hydrogen-like transitions  $\lambda$  are then given by for the outer electron

$$\underline{\frac{1}{\lambda} = R_H \cdot Z_{\text{eff}}^2 \left[ -\frac{1}{n_i^2} + \frac{1}{n_f^2} \right]}$$

For the  $1s3p \rightarrow 1s2s$  transition  $n_i = 3$ ,  $n_f = 2$

$$\frac{1}{\lambda} = 1.097 \times 10^7 \text{ [m}^{-1}\text{]} \cdot 1.8 \cdot \left[ \frac{1}{4} - \frac{1}{9} \right]$$

$$\underline{\frac{1}{\lambda} = 0.274 \times 10^7 \text{ m}^{-1}}$$

$$\boxed{\lambda = 364 \text{ nm.}}$$

ANS 5

First excited state of Be:  $1s^2 2s^1 2p^1$  (g.s. is  $1s^2 2s^2$ )

1st electron  $n, l, m_s = 2, 0, \frac{1}{2}$

2nd electron  $n, l, m_s = 2, 1, \pm \frac{1}{2}$

$\Rightarrow L = 1, S = 0, 1$

for  $S=0, L=1, J=1 \Rightarrow {}^1P_1$  state

for  $S=1, L=1, J=2, 1, 0 \Rightarrow {}^3P_{0,1,2}$

of these  ${}^3P_0$  is the lowest

First excited state of B:  $1s^2 2s^1 2p^2$  (g.s. is  $1s^2 2s^2 2p^1$ )

Step-1: work out the  $L, S$  values for  $2p^2$

$l_1 = 1, l_2 = 1 \Rightarrow L' = 2, 1, 0$

$m_{s_1} = \frac{1}{2}, m_{s_2} = \pm \frac{1}{2} \Rightarrow S' = 0, 1$

Step-2: add the  $2s^1$  electron to the above

$l = 0, L' = 2, 1, 0 \Rightarrow L = 2, 1, 0$

$s = \frac{1}{2}, S' = 0, 1 \Rightarrow S = \frac{1}{2}, \frac{3}{2}$

Step-3: pick the correct combinations

$L=0 \Rightarrow S = \frac{1}{2}$  [but not  $S = \frac{3}{2}$ ]  $\Rightarrow 2S_{1/2}$  state

$L=1 \Rightarrow \left[ \begin{array}{l} S = \frac{3}{2} \text{ but not } S = \frac{1}{2} \\ S = \frac{1}{2} \end{array} \right. \Rightarrow \begin{array}{l} 4P_{1/2} \quad 4P_{3/2} \quad 4P_{5/2} \text{ states} \\ 2P_{1/2} \quad 2P_{3/2} \end{array}$

$L=2 \Rightarrow S = \frac{1}{2}$  [but not  $S = \frac{3}{2}$ ]  $\Rightarrow 2D_{3/2} \quad 2D_{5/2}$

of these  $4P_{1/2}$  is the lowest

Q6 **ANS 6**

The system is in a mix of  $n=1$  &  $n=2$  states  
Hydrogenic energies are

$$E_{100} = -13.6 \text{ eV}$$

$$E_{200, 210, 21\pm 1} = -3.4 \text{ eV}$$

Photon energy is  $12 \text{ eV}$ , and is inadequate to excite a  $|100\rangle$  state electron to the continuum. But electrons corresponding to  $|210\rangle$ ,  $|21\pm 1\rangle$  can be excited to the continuum, and will have a K.E. of  $12 - 3.4$

$$\text{KE} = 8.6 \text{ eV.}$$

electrons in  $|100\rangle$  can get excited to other states

e.g.  $n=3, 4$  if there is an approximate energy match

e.g.  $n=3 \Rightarrow E = -1.5 \text{ eV}$  so  $12 \text{ eV}$  photons can

lead to a weak excitation (considering natural linewidths, fine structure & thermal broadening) in a large collection of such atoms.

ANS 7

$$\text{Level lifetime} = 12 \text{ ns}$$

$$\text{Line spread} = \frac{1}{2\tau} = \underline{8.3 \times 10^7 \text{ Hz}}$$

$$\text{observed } \tilde{\lambda} = 486.1 \text{ nm} \quad \Delta\lambda = 0.2 \text{ nm}$$

$$\Rightarrow \tilde{\omega} = \frac{2\pi c}{\tilde{\lambda}} = 3.87 \times 10^{15} \text{ Hz}$$

$$\frac{\Delta\omega}{\tilde{\omega}} = \frac{\Delta\lambda}{\tilde{\lambda}} ; \text{ Hence } \underline{\Delta\omega = 1.59 \times 10^{12} \text{ Hz}}$$

Note that  $\Delta\omega_{\text{obs}} \gg \Delta\omega_{\text{lifetime}}$  hence we can

approximate  $\Delta\omega_{\text{obs}} \approx \Delta\omega_{\text{thermal}}$

(Exact: the two add in quadrature)

$$\frac{\Delta\omega_{\text{th}}}{\tilde{\omega}} = \left[ \frac{k_B T}{mc^2} \right]^{1/2} \cdot (8 \ln 2)^{1/2}$$

$$\Rightarrow T \approx \underline{3 \times 10^5 \text{ Kelvin.}}$$

ANS 8

The stimulated emission increment =  $R_{21} - R_{12}$   
(excess)

$$R_{\text{incr}} = (N_2 - N_1) B_{21} u(\omega) a(\omega)$$

$$= (N_2 - N_1) B_{21} \frac{I(\omega)}{c} a(\omega)$$

For each transition (st.em) a photon of energy  $h\omega$  is added to the (laser) beam.

The energy added per unit time per unit volume of the beam is  $R_{\text{incr}} \cdot h\omega$ ; for a volume  $dV$  this will be  $R_{\text{incr}} h\omega dV$

The small increment in the light beam intensity in a length  $dx$  of a medium of cross section  $A$  will be given

by

$$dI \cdot A = R_{\text{incr}} h\omega dV$$

$$\uparrow$$

intensity is "per unit area"

$$= R_{\text{incr}} h\omega A dx$$

$$\therefore dI = (N_2 - N_1) B_{21} \frac{I(\omega)}{c} a(\omega) \cdot h\omega \cdot dx$$

There was a conflict of notation for the gain coeff in the notes and the question

(a)  $\frac{dI}{dx} = g(\omega) I(\omega)$  HERE

(b)  $g(\omega) = I(2L) / I(0)$  NOTES

If you chose (b) instead of (a) you will get something similar to this, but

in the exponent.

BOTH have been taken to be valid.

$g(\omega)$  can be written in terms of the 'A' coeff [or  $\tau$ ] instead of the 'B' coeff, since

$$A = \frac{8\pi h \nu^3}{c^3} B \quad \text{and} \quad A = \frac{1}{\tau}$$

Plugging these values into  $g(\omega)$  we get

$$g(\omega) = (N_2 - N_1) \frac{c^2 \pi}{2\omega^2 \tau} a(\omega)$$